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NUMERICAL ANALYSIS OF THIN SHELL PROBLEMS : APPROXIMATION AND SHAPE OPTIMIZATION

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ANALYSE NUMERIQUE DE PROBLEMES DE COQUES MINCES : APPROXIMATION ET OPTIMISATION DE FORME (*)

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RESUME : Ce rapport constitue une synthèse de différents résultats d'analyse numérique de problèmes de coques minces. La première partie comporte une définition géométrique générale des coques minces illustrée par différents exemples concrets. On rappelle ensuite la formulation mécanique et mathématique des problèmes et l'on donne quelques résultats d'existence.

La seconde partie du rapport fait le point sur les résultats (obtenus ou en cours de développement) relatifs à l'approximation de ces problèmes par des méthodes d'éléments finis. Ce sont essentiellement les méthodes conformes, les méthodes d'élément finis à facettes planes, les méthodes mixtes et les méthodes D.K.T.

Enfin la troisième partie rassemble les résultats de base pour aborder le problème d'optimisation de forme d'une coque mince (travail conjoint avec F. PALMA et B. ROUSSELET).

ABSTRACT : This report constitutes a synthesis of different results of numerical analysis of thin shell problems. The first part involves a general geometric definition of thin shells which is illustrated by different concrete examples. Next, one records the mechanical and the mathematical formulations of the problems and one gives some existence results.

The second part presents some results (obtained or under study) concerning the approximation of these problems by finite element methods. They include conforming methods, flat plate element methods, mixed methods and D.K.T. methods.

Finally, the third part contains basic results to solve the optimal design problem of a thin shell (joint work with F. PALMA and B. ROUSSELET).

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NUMERICAL ANALYSIS OF THIN SHELL PROBLEMS : APPROXIMATION AND SHAPE OPTIMIZATION

PART I : GENERAL PRESENTATION OF THIN SHELL EQUATIONS

1 - INTRODUCTION

A shell is a three-dimensional continuous medium for which one dimension, the thickness, is "small" with respect to the two others. Under the action of sufficiently small loads, the shell, initially unconstrained, is deformed following the usual laws of the three-dimensional elasticity. The basic idea of a *first family of shell theories* is to take into account the particular geometry of such a medium and, by "*integration through the thickness*" to obtain a two-dimensional model, formulated in terms of the *middle surface of the shell*, which represents a "good" approximation of the three-dimensional model.

The pioneers of this kind of derivation are KIRCHHOFF [1876] and LOVE [1934]. Their theories were developed and improved by numerous authors, especially by KOITER [1966,1970] and KOITER and SIMMONDS [1973]. The mathematical analysis of such derivation methods is now in progress, especially with the works of CIARLET and DESTUYNDER [1979] for plate problems and DESTUYNDER [1980] for shell problems.

A second very well-known family of shell theories is based on the COSSERAT [1909] surface theory ; it has been developed by NAGHDI [1963,1972] among others. Though the basic ideas of these two families are different, their numerical analysis is very similar.

Subsequently, we will use the KOITER model, which originates from the displacement formulation of three-dimensional elasticity. By means of suitable assumptions about the types of loads applied to the shell and on stress distribution, KOITER has obtained a two-dimensional formulation in terms of geometrical properties of the middle surface of the shell, for which the unknown is *the displacement field of the particles comprising the middle surface*. From the knowledge of this displacement field, one deduces the displacement field and the stress field for any particle of the shell.

In the following, we will consider

i) some general formulations of thin shell problems according to KOITER's models (Part I) ;

ii) some different ways to approximate the solutions of linear equations by using finite element methods (Part II) ;

iii) a convenient way to obtain the derivatives of some functionals to be minimized in view of optimizing *the design of a shell* : the *design variables* are the parametric functions defining the *middle surface* and the *thickness* of the shell (This part III is a joint work with F. PALMA and B. ROUSSELET).

CONTENTS

PART III - OPTIMAL DESIGN OF A SHELL : COMPUTATION OF THE FUNCTIONAL DERIVATIVE

Some notations

Throughout this paper, we shall frequently make use of the properties of the *SOBOLEV* spaces. Let Ω be an open bounded subset in a plane $\mathscr{E}^2$. Then, we set

$$\left.\begin{array}{l} W^{m,p}(\Omega) = \{v \in L^p(\Omega) \ : \ D^\alpha v \in L^p(\Omega) \text{ for } |\alpha| \le m\} \\ m \ge 1 \text{ integer}, \ 1 \le p < \infty \ , \end{array}\right\}$$

with the usual extension to the case $p = +\infty$. When equipped with the norm

$$\|v\|_{m,p,\Omega} = \Big(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha v|^p \, d\xi^1 d\xi^2 \ \Big)^{1/p}$$

$W^{m,p}(\Omega)$ is a BANACH space. Here ξ^1,ξ^2 denote a system of orthonormal coordinates of the $\mathscr{E}^2$-plane. The corresponding semi-norm is

$$|v|_{m,p,\Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v|^p \, d\xi^1 d\xi^2 \ \Big)^{1/p} .$$

In the following, for the case $p = 2$, we shall write $\|v\|_{m,\Omega}$ and $|v|_{m,\Omega}$ instead of $\|v\|_{m,2,\Omega}$ and $|v|_{m,2,\Omega}$. In particular, the space $W^{m,2}(\Omega) = H^m(\Omega)$ is a HILBERT space when endowed with the scalar product

$$((u,v))_{m,\Omega} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u \, D^\alpha v \, d\xi^1 d\xi^2 .$$

In view of section 2.4, we record here some basic properties of the SOBOLEV spaces. The notation $X \hookrightarrow Y$ indicates that the normed linear space X is contained into the normed linear space Y with a continuous injection. By *the SOBOLEV's imbedding theorems*, the following inclusions hold, for all integers $m \ge 0$ and all $1 \le p \le \infty$:

$$W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n} \quad , \quad \text{if } m < \frac{n}{p}$$

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1,\infty[\ , \quad \text{if } m = \frac{n}{p} \ ,$$

$$W^{m,p}(\Omega) \hookrightarrow \mathscr{C}(\bar{\Omega}) \quad \text{if } \frac{n}{p} < m \ .$$

For more details on SOBOLEV spaces, we refer to ADAMS [1975], LIONS and MAGENES [1968], NECAS [1967] and ODEN and REDDY [1976].

2 - GEOMETRICAL DEFINITION OF THE UNDEFORMED SHELL $\mathscr{C}$

Orientation : W.T. KOITER's linear theory of thin elastic shells makes use of intrinsic geometrical properties of the middle surface of the undeformed shell. Our first task is to define this middle surface and to record relevant results on differential geometry needed in the following. Next, from the knowledge of the geometry of the middle surface, we give the definition of the undeformed shell.

2.1. Definition of the middle surface

Let $\mathscr{E}^3$ be the usual euclidean space referred to an *orthonormal fixed system* $(0,\vec{e}_1,\vec{e}_2,\vec{e}_3)$, and let Ω be a bounded open subset in a plane $\mathscr{E}^2$, with a boundary Γ. Then, the *middle surface* $\bar{S}$ of the shell is the image in $\mathscr{E}^3$ of the set $\bar{\Omega}$ by the mapping $\vec{\phi}$:

$$\vec{\phi} : (\xi^1,\xi^2) \in \bar{\Omega} \subset \mathscr{E}^2 \rightarrow \vec{\phi}(\xi^1,\xi^2) \in \mathscr{E}^3 \quad . \tag{2.1.1}$$

We denote $\partial S = \vec{\phi}(\Gamma)$, hence $\bar{S} = S \cup \partial S$, and we assume $\vec{\phi}$ and Γ sufficiently smooth. Particularly, we assume that all the points of the surface $\bar{S} = \vec{\phi}(\bar{\Omega})$ are *regular* so that the vectors

$$\vec{a}_\alpha = \vec{\phi}_{,\alpha} = \frac{\partial\vec{\phi}}{\partial\xi^\alpha} \quad , \quad \alpha = 1,2, \tag{2.1.2}$$

are linearly independent for all points $\xi = (\xi^1,\xi^2) \in \bar{\Omega}$. These two vectors define the *tangent plane* to the surface $\bar{S}$ at the point $\vec{\phi}(\xi)$. Next, we define the *normal vector*

$$\vec{a}_3 = \frac{\vec{a}_1\times\vec{a}_2}{|\vec{a}_1\times\vec{a}_2|} \quad , \tag{2.1.3}$$

$|.|$ denoting the euclidean norm in $\mathscr{E}^3$ equipped with its usual scalar product $(\vec{a},\vec{b}) \rightarrow \vec{a}.\vec{b}$. Then, the point $\vec{\phi}(\xi)$ and the three vectors $\vec{a}_i$ define a *local reference system* for the middle surface (see Figure 2.1.1), i.e., the *covariant basis* attached to the point $\vec{\phi}(\xi)$.

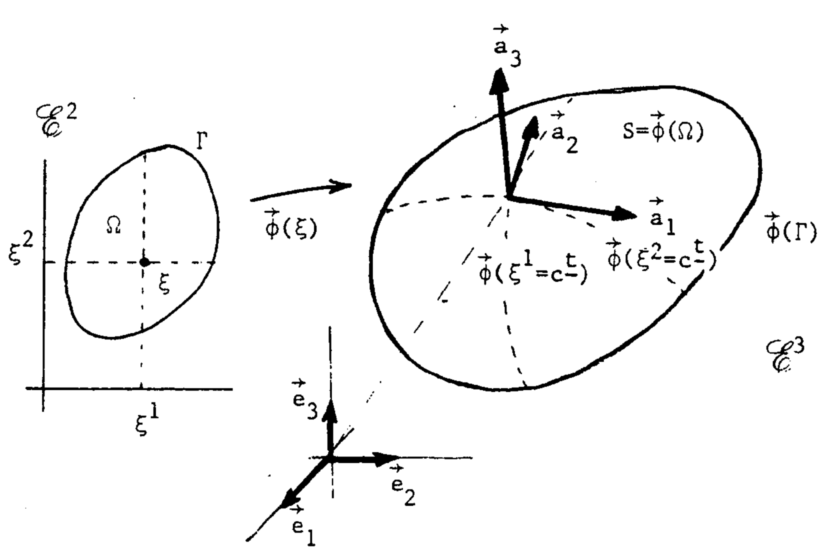


Figure 2.1.1 : Definition of the middle surface $\bar{S}$

In the following, we denote by $a_{\alpha\beta}$, $b_{\alpha\beta}$ the *first and second fundamental forms* of the middle surface S ; that is,

$$a_{\alpha\beta} = a_{\beta\alpha} = \vec{a}_\alpha.\vec{a}_\beta = \vec{\phi}_{,\alpha}.\vec{\phi}_{,\beta} \quad , \tag{2.1.4}$$

$$b_{\alpha\beta} = b_{\beta\alpha} = - \vec{a}_\alpha.\vec{a}_{3,\beta} = \vec{a}_3.\vec{a}_{\alpha,\beta} = \vec{a}_3.\vec{a}_{\beta,\alpha} \quad . \tag{2.1.5}$$

As a rule, we shall use Greek letters, $\alpha,\beta,\ldots$, for indices which take their values in the set {1,2} ; Latin letters, $i,j,\ldots$, will be used for indices that take their values in the set {1,2,3}. In addition, we shall employ the summation convention for a repeated index, occuring once as a subscript and once as a superscript.

To the vectors $\vec{a}_\alpha$, we associate two other vectors $\vec{a}^\beta$ of the tangent plane defined by

$$\vec{a}_\alpha.\vec{a}^\beta = \delta^\beta_\alpha \quad , \quad \delta^\beta_\alpha = \begin{cases} 1 \text{ if } \alpha = \beta \quad , \\ 0 \text{ if } \alpha \neq \beta \quad , \end{cases} \tag{2.1.6}$$

These vectors are linked to the vectors $\vec{a}_\alpha$ by the relations

$$\vec{a}_\alpha = a_{\alpha\beta}\vec{a}^\beta \quad , \quad \vec{a}^\alpha = a^{\alpha\beta}\vec{a}_\beta \quad , \quad a^{\alpha\beta} = \vec{a}^\alpha.\vec{a}^\beta = a^{\beta\alpha} \quad , \tag{2.1.7}$$

where the matrix $(a^{\alpha\beta})$ is the inverse of the matrix $(a_{\alpha\beta})$. This inverse matrix is well defined, since all the points of the middle surface S are assumed to be regular. The set $(\vec{a}^1,\vec{a}^2,\vec{a}^3)$ defines the *contravariant basis* attached to the point $\vec{\phi}(\xi)$.

For a given tensor, the metric tensors $(a_{\alpha\beta})$ and $(a^{\alpha\beta})$ permit us to derive the different kinds of components. For instance, to the *covariant components* $b_{\alpha\beta}$ of the second fundamental form, we can associate the following *mixed and contravariant components*

$$\left.\begin{aligned} & b_\alpha^{\cdot\beta} = b^\beta_{\cdot\alpha} = b^\beta_\alpha = a^{\beta\lambda}b_{\lambda\alpha} \quad , \\ & b^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\nu}b_{\lambda\nu} \quad , \end{aligned}\right\} \tag{2.1.8}$$

and, conversely,

$$b_{\alpha\beta} = a_{\alpha\lambda}b^\lambda_\beta = a_{\alpha\lambda}a_{\beta\nu}b^{\lambda\nu} \quad . \tag{2.1.9}$$

Since the basis $(\vec{a}_1,\vec{a}_2,\vec{a}_3)$ and $(\vec{a}^1,\vec{a}^2,\vec{a}^3)$ are neither normed nor orthogonal, it is somewhat complicated to compute their derivatives. Thus, it is convenient to introduce the *CHRISTOFFEL's symbols* $\Gamma^\alpha_{\beta\gamma}$ defined by

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \vec{a}^\alpha.\vec{a}_{\gamma,\beta} = \vec{a}^\alpha.\vec{a}_{\beta,\gamma} \quad , \tag{2.1.10}$$

as well as the notion of *covariant derivatives* for a surface tensor. For instance, for tensors of order 1 or 2, we have :

$$\left.\begin{aligned} T_{\alpha|\gamma} &= T_{\alpha,\gamma} - \Gamma^{\lambda}_{\alpha\gamma} T_{\lambda} \quad , \\ T^{\alpha}|_{\gamma} &= T^{\alpha}_{,\gamma} + \Gamma^{\alpha}_{\lambda\gamma} T^{\lambda} \quad , \end{aligned}\right\} \tag{2.1.11}$$

$$\left.\begin{aligned} T_{\alpha\beta|\gamma} &= T_{\alpha\beta,\gamma} - \Gamma^{\lambda}_{\alpha\gamma} T_{\lambda\beta} - \Gamma^{\lambda}_{\beta\gamma} T_{\alpha\lambda} \quad , \\ T^{\alpha}_{\cdot\beta|\gamma} &= T^{\alpha}_{\cdot\beta,\gamma} + \Gamma^{\alpha}_{\gamma\lambda} T^{\lambda}_{\cdot\beta} - \Gamma^{\lambda}_{\beta\gamma} T^{\alpha}_{\cdot\lambda} \quad , \\ T^{\alpha\beta}|_{\gamma} &= T^{\alpha\beta}_{\quad,\gamma} + \Gamma^{\alpha}_{\lambda\gamma} T^{\lambda\beta} + \Gamma^{\beta}_{\lambda\gamma} T^{\alpha\lambda} \quad . \end{aligned}\right\} \tag{2.1.12}$$

Relations (2.1.5) and (2.1.10) involve

$$\left.\begin{aligned} \vec{a}_{\alpha,\beta} &= \Gamma^{\gamma}_{\alpha\beta} \vec{a}_{\gamma} + b_{\alpha\beta} \vec{a}_3 \\ \vec{a}^{\alpha}_{,\beta} &= - \Gamma^{\alpha}_{\beta\lambda} \vec{a}^{\lambda} + b^{\alpha}_{\beta} \vec{a}_3 \end{aligned}\right\} \qquad \text{(GAUSS)} \tag{2.1.13}$$

$$\vec{a}_{3,\alpha} = \vec{a}^{3}_{,\alpha} = - b^{\gamma}_{\alpha} \vec{a}_{\gamma} \qquad \text{(WEINGARTEN)} \tag{2.1.14}$$

Expressions of some cross products of basis vectors are also worth noting :

$$\left.\begin{aligned} \vec{a}_{\alpha} \times \vec{a}_{\beta} &= \epsilon_{\alpha\beta} \vec{a}^{3} \quad , \\ \vec{a}^{\alpha} \times \vec{a}^{\beta} &= \epsilon^{\alpha\beta} \vec{a}_{3} \quad , \\ \vec{a}_{3} \times \vec{a}_{\beta} &= \epsilon_{\beta\lambda} \vec{a}^{\lambda} \quad , \\ \vec{a}_{3} \times \vec{a}^{\beta} &= \epsilon^{\beta\lambda} \vec{a}_{\lambda} \quad . \end{aligned}\right\} \tag{2.1.15}$$

Here,

$$\epsilon_{\alpha\beta} = \sqrt{a}\; e_{\alpha\beta} \quad , \quad \epsilon^{\alpha\beta} = \frac{1}{\sqrt{a}}\; e^{\alpha\beta} \quad , \tag{2.1.16}$$

$$(e_{\alpha\beta}) = (e^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.1.17}$$

$$a = a_{11} a_{22} - (a_{12})^2 \neq 0 \text{ (regular points)} \quad . \tag{2.1.18}$$

The parameter a appears in the expression of the *area element* dS *of the surface*, that is,

$$dS = |\vec{a}_1 \times \vec{a}_2|\; d\xi^1 d\xi^2 = \sqrt{a}\; d\xi^1 d\xi^2 \quad . \tag{2.1.19}$$

2.2. Geometrical definition of the undeformed shell $\mathscr{C}$

In addition to the two curvilinear coordinates ξ^1, ξ^2 that allow definition of the middle surface, we introduce a third curvilinear coordinate, ξ^3, which is measured along the normal $\vec{a}_3$ to the surface $\bar{S}$ at point $\vec{\phi}(\xi^1, \xi^2)$. This system (ξ^1, ξ^2, ξ^3) of curvilinear coordinates is, at least locally, a system of curvilinear coordinates of $\mathscr{E}^3$, generally called *normal coordinates system*.

The *thickness* e of the shell is defined through a mapping

$$e : (\xi^1, \xi^2) \in \bar{\Omega} \to \{x \in \mathbb{R} \; ; \; x > 0\} \quad . \tag{2.2.1}$$

Then, the *shell* $\mathscr{C}$ is the closed subset of $\mathscr{E}^3$ defined by

$$\left.\begin{array}{l} \mathscr{C} = \{M \in \mathscr{E}^3 \; ; \; \vec{OM} = \vec{\phi}(\xi^1,\xi^2) + \xi^3 \vec{a}_3 \; , \; (\xi^1,\xi^2) \in \bar{\Omega} \; , \\ \qquad\qquad - \frac{1}{2} e(\xi^1,\xi^2) \le \xi^3 \le \frac{1}{2} e(\xi^1,\xi^2)\} \quad . \end{array}\right\} \tag{2.2.2}$$

The derivatives of the vector $\vec{OM} = \vec{\phi}(\xi^1,\xi^2) + \xi^3 \vec{a}_3$ are vectors $\vec{g}_i$ which, by virtue of (2.1.14) satisfy

$$\left.\begin{array}{l} \vec{g}_\alpha = \vec{OM}_{,\alpha} = (\delta^\nu_\alpha - \xi^3 b^\nu_\alpha) \vec{a}_\nu \quad , \\ \vec{g}_3 = \vec{OM}_{,3} = \vec{a}_3 \quad . \end{array}\right\} \tag{2.2.3}$$

The vectors $\vec{g}_1$ and $\vec{g}_2$ are parallel to the tangent plane to the middle surface at point $\vec{\phi}(\xi^1,\xi^2)$ while vector $\vec{g}_3$ is normal to this plane (see Figure 2.2.1). In BERNADOU and CIARLET [1976, § 2.1] it is shown that $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$ is a *local reference system* at any point of the shell $\mathscr{C}$.

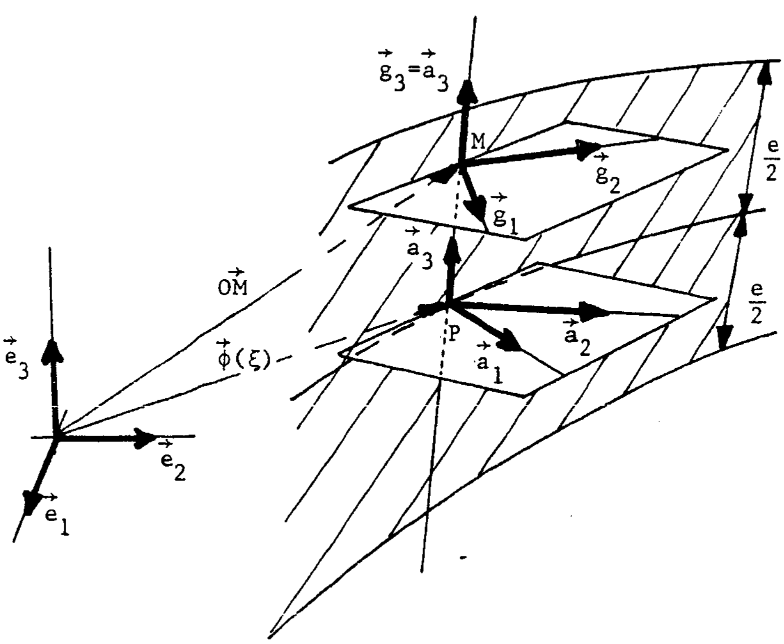


Figure 2.2.1 : Different local basis through the thickness

2.3. Some examples

2.3.1. Cylindrical roof

R = 300 in
L = 600 in
e = 3.0 in (thickness)
E = 3.0 × 10^6 psi
ν = 0.0
ϕ = 40°

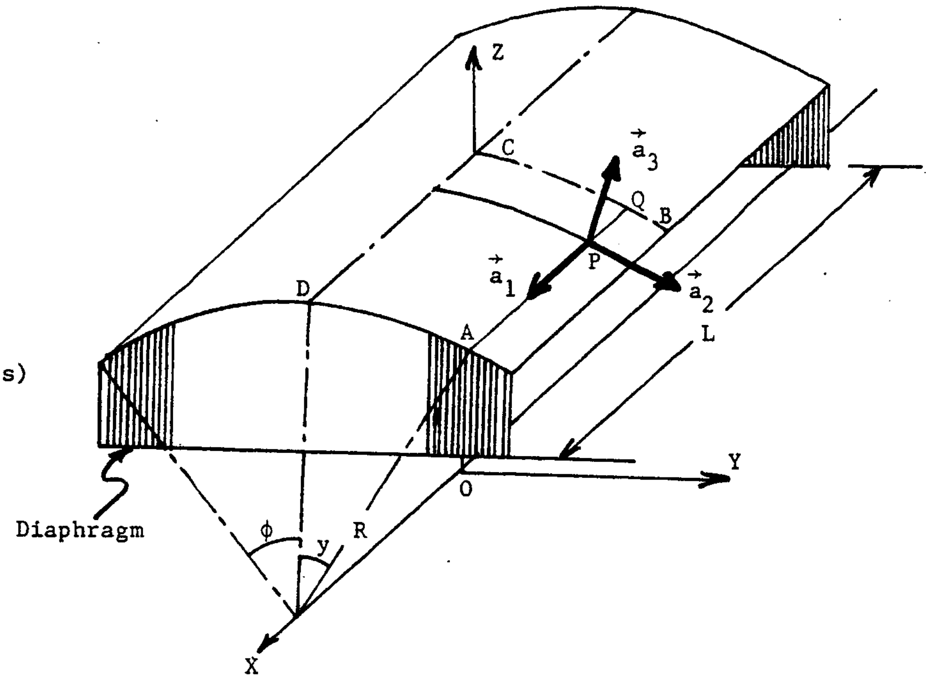


Figure 2.3.1 : The cylindrical roof

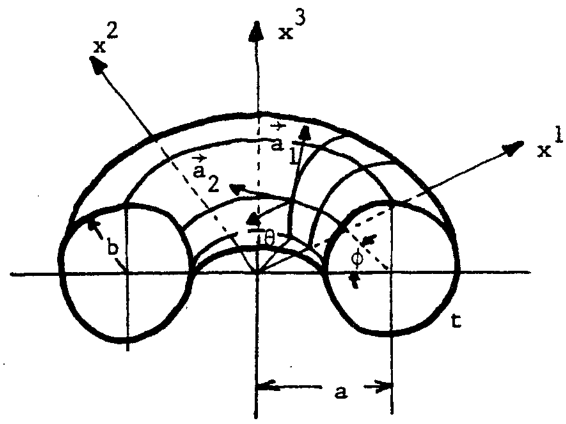
$$\vec{OP} = \vec{\phi}(x,y) = \begin{cases} x \\ R \sin y \\ R \cos y \end{cases} \qquad (2.3.1)$$

where $\xi^1 = x$, $\xi^2 = y = (\vec{OQ},\vec{OC})_{\vec{OX}}$.

From this geometrical representation of the middle surface, we obtain the following geometrical characteristics :

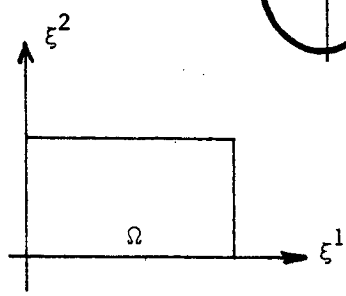
$$\left.\begin{array}{l}
\vec{a}_1 = \begin{cases} 1 \\ 0 \\ 0 \end{cases} \quad ; \qquad \vec{a}_2 = \begin{cases} 0 \\ R \cos y \\ -R \sin y \end{cases} \quad ; \qquad \vec{a}_3 = \begin{cases} 0 \\ \sin y \\ \cos y \end{cases} \\
a_{11} = 1 \quad ; \quad a_{12} = a_{21} = 0 \quad ; \quad a_{22} = R^2 \quad ; \quad a = R^2 \quad ; \\
a^{11} = 1 \quad ; \quad a^{12} = a^{21} = 0 \quad ; \quad a^{22} = \frac{1}{R^2} \\
\vec{a}^1 = \vec{a}_1 \quad ; \quad \vec{a}^2 = \frac{1}{R^2}\vec{a}_2 \quad ; \quad \vec{a}^3 = \vec{a}_3 \\
b_{11} = b_{12} = b_{21} = 0 \quad ; \quad b_{22} = -R \\
b_1^1 = b_2^1 = b_1^2 = 0 \quad ; \quad b_2^2 = -\frac{1}{R} \\
b^{11} = b^{12} = b^{21} = 0 \quad ; \quad b^{22} = -\frac{1}{R^3} \\
\Gamma^{\alpha}_{\beta\lambda} = 0
\end{array}\right\} \qquad (2.3.2)$$

2.3.2. <u>Torus</u>



$a = 15$
$b = 10$
$t = 0.5$

$$\begin{cases} \xi^1 = \phi \\ \xi^2 = \theta \end{cases}$$

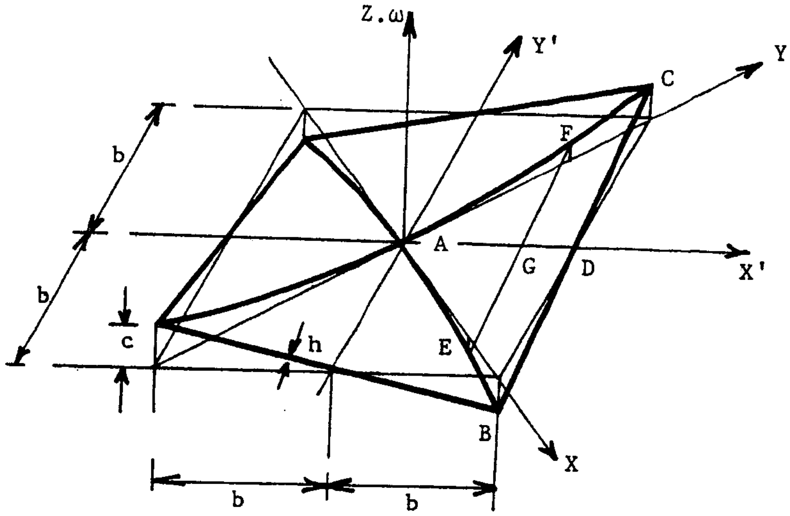


<u>Figure 2.3.2</u> : <u>The torus</u>

$$x(\xi^1,\xi^2) = x^i(\xi^1,\xi^2)\vec{e}_i = \begin{cases} (a\text{-}b \cos \xi^1) \cos \xi^2 \\ (a\text{-}b \cos \xi^1) \sin \xi^2 \\ b \sin \xi^1 \end{cases} \tag{2.3.3}$$

$$\left.\begin{aligned}
&\vec{a}_1 = b \begin{cases} \sin \xi^1 \cos \xi^2 \\ \sin \xi^1 \sin \xi^2 \\ \cos \xi^1 \end{cases} \quad ; \quad \vec{a}_2 = (a\text{-}b \cos \xi^1) \begin{cases} -\sin \xi^2 \\ \cos \xi^2 \\ 0 \end{cases} \quad ; \quad \vec{a}_3 = \begin{cases} -\cos \xi^1 \cos \xi^2 \\ -\cos \xi^1 \sin \xi^2 \\ \sin \xi^1 \end{cases} \\
&a_{11} = b^2 \quad ; \quad a_{12} = 0 \quad ; \quad a_{22} = (a\text{-}b \cos \xi^1)^2 \\
&a = b^2(a\text{-}b \cos \xi^1)^2 \\
&a^{11} = \frac{1}{b^2} \quad ; \quad a^{12} = 0 \quad ; \quad a^{22} = \frac{1}{(a\text{-}b \cos \xi^1)^2} \\
&\vec{a}^1 = a^{11}\vec{a}_1 + a^{12}\vec{a}_2 = \frac{1}{b} \begin{cases} \sin \xi^1 \cos \xi^2 \\ \sin \xi^1 \sin \xi^2 \\ \cos \xi^1 \end{cases} \quad ; \quad \vec{a}^2 = a^{12}\vec{a}_1 + a^{22}\vec{a}_2 = \frac{1}{a\text{-}b \cos \xi^1} \begin{cases} -\sin \xi^2 \\ \cos \xi^2 \\ 0 \end{cases} \\
&b_{11} = -\vec{a}_1 . \vec{a}_{3,1} = -b \begin{cases} \sin \xi^1 \cos \xi^2 \\ \sin \xi^1 \sin \xi^2 \\ \cos \xi^1 \end{cases} . \begin{cases} \sin \xi^1 \cos \xi^2 \\ \sin \xi^1 \sin \xi^2 \\ \cos \xi^1 \end{cases} = -b \quad ; \quad b_{12} = 0 \quad ; \\
&b_{22} = -\vec{a}_2 . \vec{a}_{3,2} = -(a\text{-}b \cos \xi^1) \begin{cases} -\sin \xi^2 \\ \cos \xi^2 \\ 0 \end{cases} . \begin{cases} + \cos \xi^1 \sin \xi^2 \\ - \cos \xi^1 \cos \xi^2 \\ 0 \end{cases} = (a\text{-}b \cos \xi^1) \cos \xi^1 \\
&\Gamma^1_{11} = \vec{a}^1 . \vec{a}_{1,1} = \frac{1}{b} \begin{cases} \sin \xi^1 \cos \xi^2 \\ \sin \xi^1 \sin \xi^2 \\ \cos \xi^1 \end{cases} . \, b \begin{cases} \cos \xi^1 \cos \xi^2 \\ \cos \xi^1 \sin \xi^2 \\ -\sin \xi^1 \end{cases} = 0 \quad ; \quad \text{etc...}
\end{aligned}\right\} \tag{2.3.4}$$

2.3.3. Hyperbolic paraboloid



b = 50 cm
c = 10 cm
h = 0.8 cm
E = 2.85.10⁴ kp/cm²
ν = 0.4
q = 0.01 kp/cm²

Figure 2.3.3 : The hyperbolic paraboloid

$$\vec{OP} = \vec{\phi}(x,y) = \begin{cases} x(=X = \xi^1) \\ y(=Y = \xi^2) \\ \frac{c}{2b^2}(y^2 - x^2) \end{cases} \tag{2.3.5}$$

$$\left.\begin{aligned} &\vec{a}_1 = \begin{cases} 1 \\ 0 \\ -\frac{c}{b^2}x \end{cases} \quad ; \quad \vec{a}_2 = \begin{cases} 0 \\ 1 \\ \frac{c}{b^2}y \end{cases} \quad ; \quad \vec{a}_3 = \frac{1}{\sqrt{1+\frac{c^2}{b^4}(x^2+y^2)}} \begin{cases} -\frac{c}{b^2}x \\ -\frac{c}{b^2}y \\ 1 \end{cases} \\ &a_{11} = 1 + \frac{c^2}{b^4}x^2 \quad ; \quad a_{12} = a_{21} = -\frac{c^2}{b^4}xy \quad ; \quad a_{22} = 1 + \frac{c^2}{b^4}y^2 \\ &a = a_{11}a_{22} - (a_{12})^2 = 1 + \frac{c^2}{b^4}(x^2+y^2) \end{aligned}\right\} \tag{2.3.6}$$

$$\left.\begin{aligned} &a^{11} = \frac{1}{a}(1+\frac{c^2}{b^4}y^2) \quad ; \quad a^{12} = a^{21} = \frac{c^2}{ab^4}xy \quad ; \quad a^{22} = \frac{1}{a}(1+\frac{c^2}{b^4}x^2) \\ &\vec{a}_1 = \frac{1}{a}\begin{cases} 1+\frac{c^2}{b^4}y^2 \\ \frac{c^2}{b^4}xy \\ -\frac{cx}{b^2} \end{cases} \quad ; \quad \vec{a}_2 = \frac{1}{a}\begin{cases} \frac{c^2}{b^4}xy \\ 1+\frac{c^2}{b^4}x^2 \\ \frac{c}{b^2}y \end{cases} \quad ; \quad \vec{a}^3 = \vec{a}_3 \end{aligned}\right\} \tag{2.3.7}$$

$$\left.\begin{array}{l}
b_{11} = -\frac{c}{b^2\sqrt{a}} \;;\; b_{12} = b_{21} = 0 \;;\; b_{22} = \frac{c}{b^2\sqrt{a}} \\
\left\{\begin{array}{l} b_1^1 = -\frac{c}{b^2 a\sqrt{a}}\left(1+\frac{c^2}{b^4}y^2\right) \;;\; b_1^2 = -\frac{c^3}{b^6 a\sqrt{a}}xy \\ b_2^1 = \frac{c^3}{b^6 a\sqrt{a}}xy \;;\; b_2^2 = \frac{c}{ab^2\sqrt{a}}\left(1+\frac{c^2}{b^4}x^2\right) \end{array}\right. \\
\left\{\begin{array}{l} b^{11} = -\frac{c}{b^2a^2\sqrt{a}}\left[1 + 2\frac{c^2}{b^4}y^2 + \frac{c^4y^2}{b^8}(y^2-x^2)\right] \\ b^{12} = b^{21} = \frac{c^5xy}{b^{10}a^2\sqrt{a}}(x^2-y^2) \\ b^{22} = \frac{c}{b^2a^2\sqrt{a}}\left[1 + 2\frac{c^2}{b^4}x^2 + \frac{c^4x^2}{b^8}(x^2-y^2)\right] \end{array}\right. \\
\left\{\begin{array}{l} \Gamma_{11}^1 = \frac{c^2x}{ab^4} \;;\; \Gamma_{11}^2 = -\frac{c^2y}{ab^4} \;;\; \Gamma_{12}^1 = \Gamma_{12}^2 = 0 \;; \\ \Gamma_{22}^1 = -\frac{c^2x}{ab^4} \;;\; \Gamma_{22}^2 = \frac{c^2y}{ab^4} \;. \end{array}\right.
\end{array}\right\} \qquad (2.3.8)$$

2.3.4. An arch dam

Figures 2.3.4 and 2.3.5 are extracted from the preliminary file of GRAND'MAISON arch dam studied by COYNE-et-BELLIER [1977]. Figure 2.3.4 gives the *site plane of the dam* with different level lines as also excavation results. In Figure 2.3.5, we have the *geometrical definition of the dam* and the horizontal sections of the upper arcs.

In these figures one can note that the definition of this arch dam is very close to the definition of a thin shell as given in (2.2.2). For simplicity, we introduce two slight modifications to the geometrical definition of the dam as proposed by COYNE-et-BELLIER :

(1) *the middle surface of the dam will be assumed symmetric* with respect to the vertical plane including the dam axis. This is not true in Figure 2.3.5 : at the heights 1697 and 1630, lateral sides are not symmetrical.

(2) *the thickness of the shell will be measured along the normal to the middle surface* in agreement with definition (2.2.2). This is only approximatively true in Figure 2.3.5 where the thickness is measured along an horizontal line.

Definition of the middle surface

The middle surface S of the arch dam is referred to a fixed orthonormal reference system $(0,\vec{e}_1,\vec{e}_2,\vec{e}_3)$ of the euclidean space $\mathscr{E}^3$ as indicated in Figure 2.3.6. The origin of this reference system is located at the intersection of the crest line with the symmetry plane of the arch dam. Vector $\vec{e}_1$ is horizontal, in the symmetry plane of the arch dam and oriented towards the upstream part of the dam. Vector $\vec{e}_3$ is vertical and oriented from top to bottom. In this reference system, the coordinates (x^1,x^2,x^3) of any point M are such that

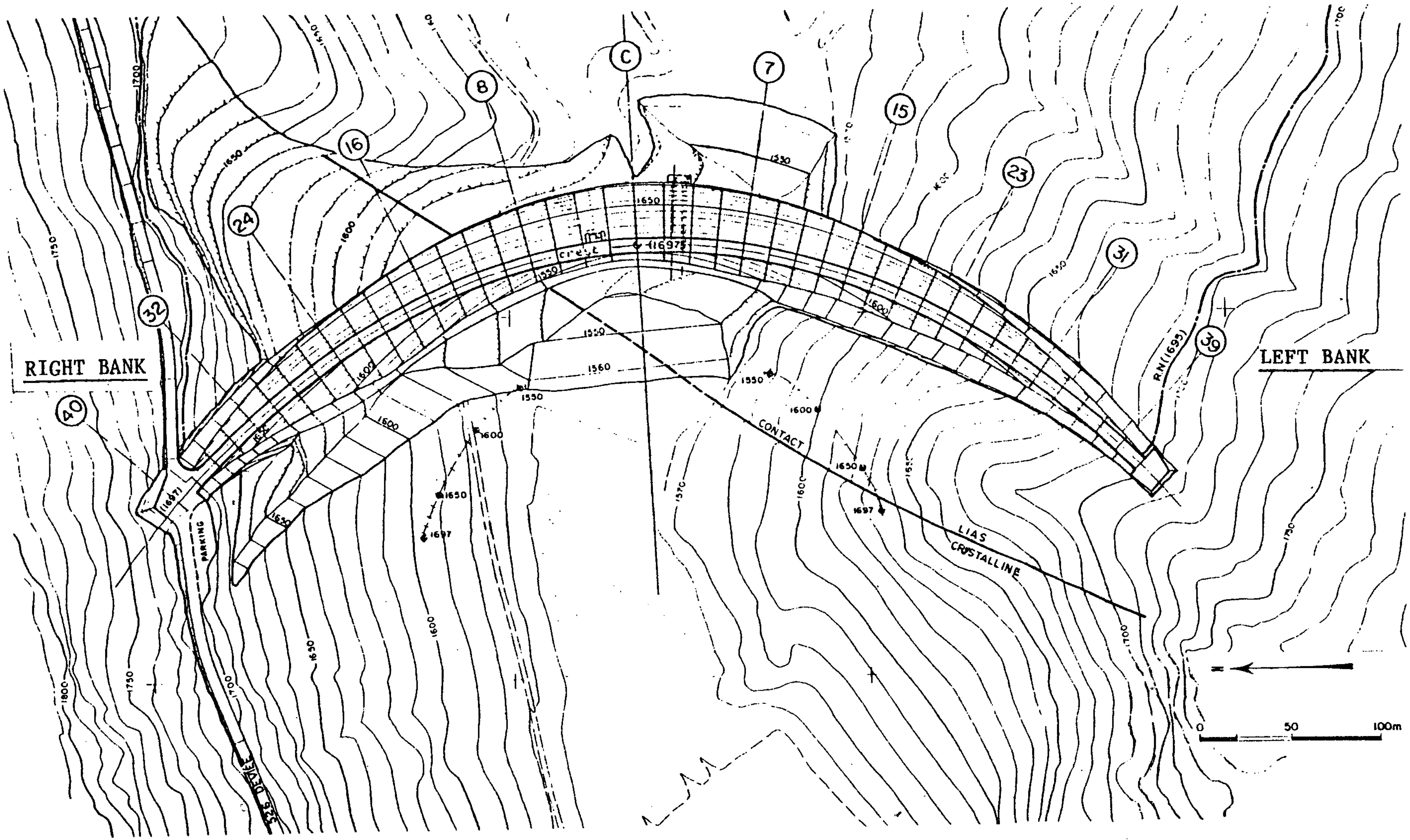


Figure 2.3.4 : Sit plane of the dam (plane view)

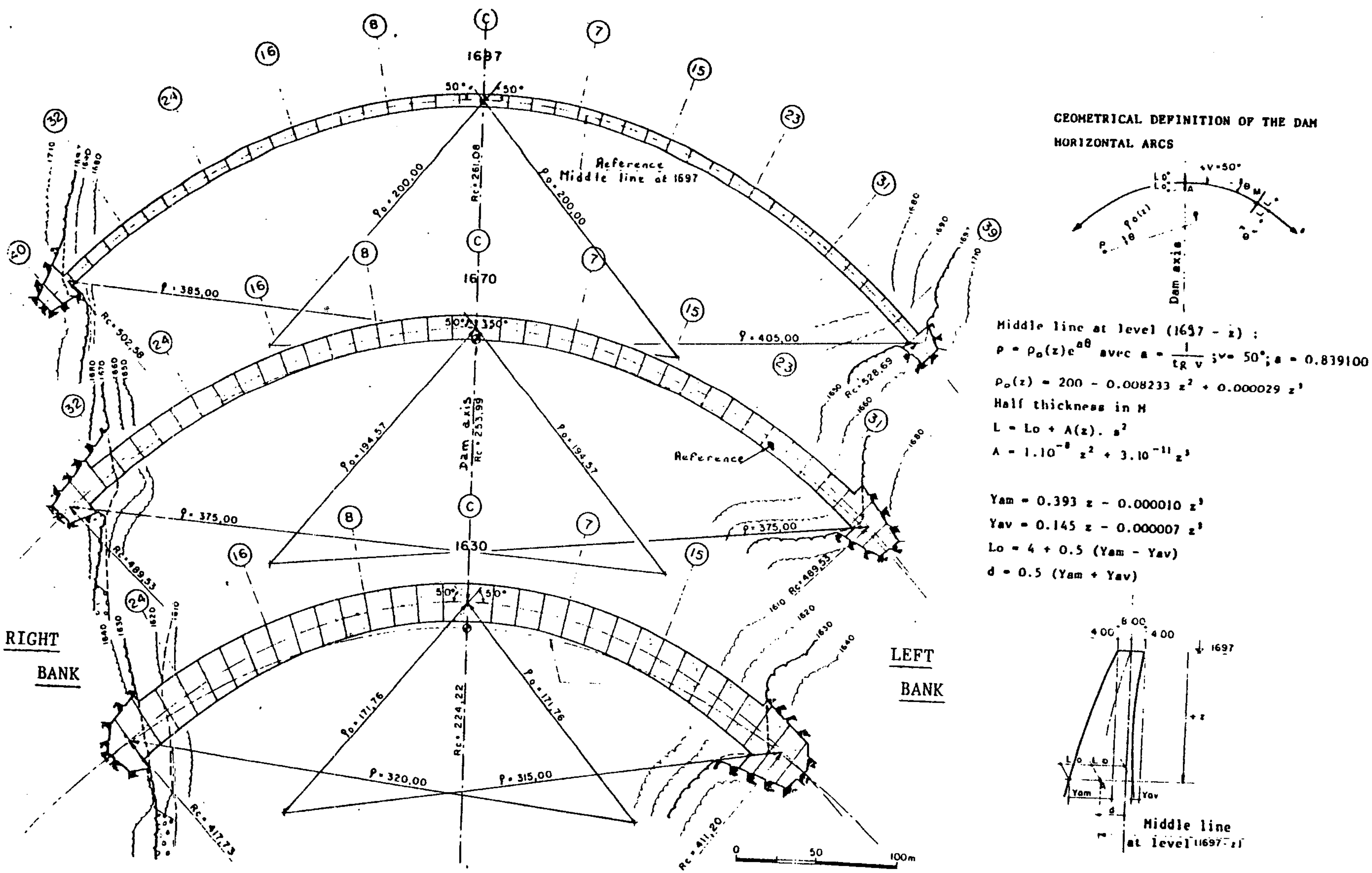


Figure 2.3.5 : Geometric definition of the dam - Upper horizontal sections

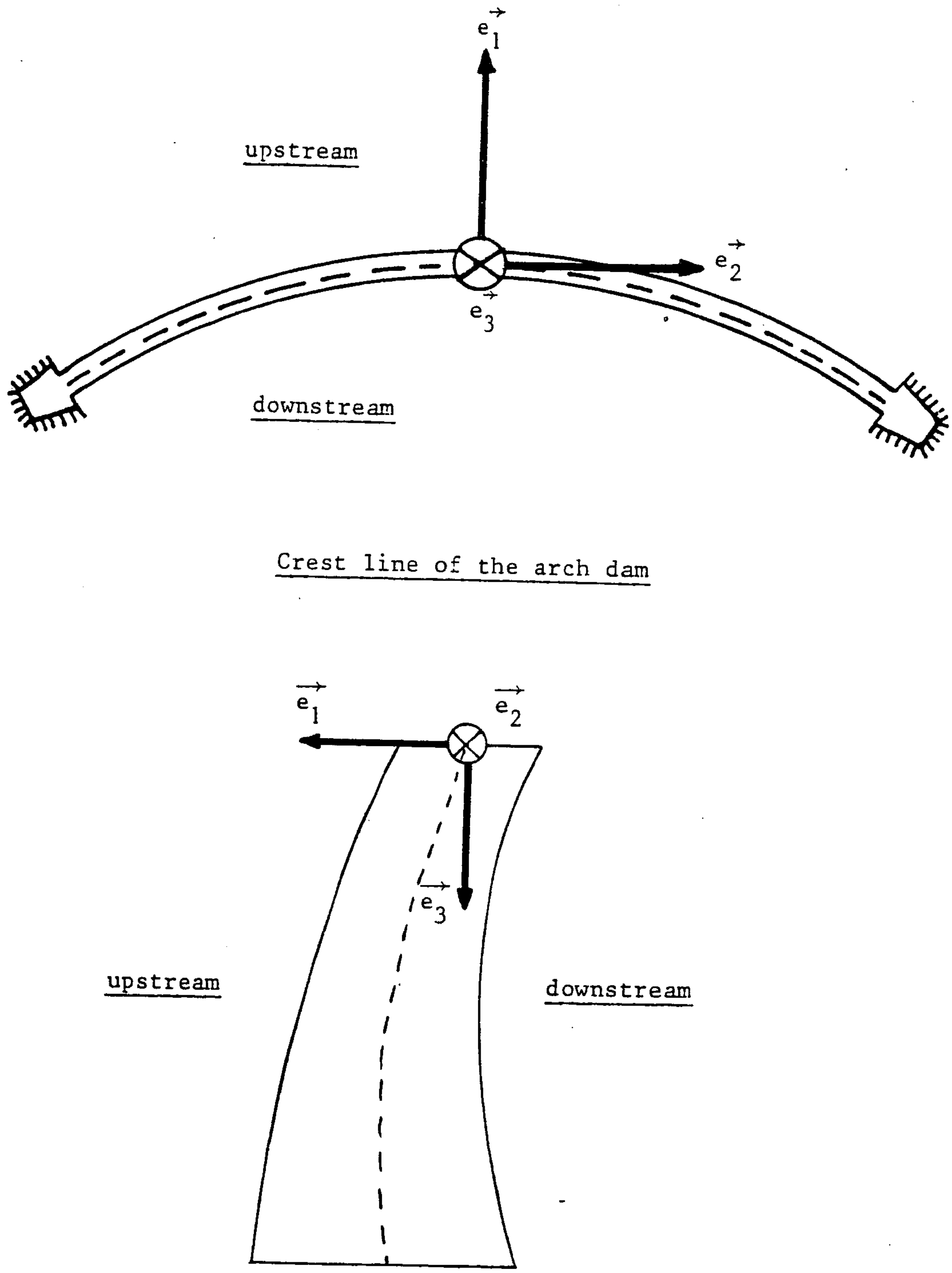


Figure 2.3.6 : The fixed orthonormal reference system of the space $\mathscr{E}^3$ (in the figure, the thickness of the dam is defined horizontally as indicated in COYNE and BELLIER [1977])

$$\vec{OP} = x^i \vec{e}_i \quad , \tag{2.3.9}$$

(using EINSTEIN's convention).

The middle surface S of the arch dam is defined as the image of a plane domain Ω through a map $\vec{\phi}$, (see Figure 2.3.7) i.e.,

$$\vec{\phi} : (\xi^1, \xi^2) \in \Omega \rightarrow \vec{OP} = \vec{\phi}(\xi^1, \xi^2) = x^i(\xi^1, \xi^2)\vec{e}_i \quad . \tag{2.3.10}$$

The curvilinear coordinates (ξ^1, ξ^2) are given by

$$(\xi^1 = \frac{\theta}{\theta_o} \quad , \quad \xi^2 = \frac{Z}{Z_o}) \quad , \tag{2.3.11}$$

where the parameters θ and Z are defined in Figure 2.3.5 and where $\theta_o = \max|\theta|$, $Z_o = \max Z$: the maximum is taken along the arch dam, for instance Z_o is the height of the arch dam.

Then, by observing Figure 2.3.5, we see that the generic point P of the middle surface S is defined by its coordinates :

$$\left.\begin{aligned}
x^1(\xi^1, \xi^2) &= \rho_o(\xi^2) \; [e^{\alpha\theta_o|\xi^1|} \cos(\theta_o|\xi^1| + 40°) - \cos 40°] \\
&\quad + 0.269 \; Z_o\xi^2 - 0.0000085 \; Z_o^3(\xi^2)^3 \\
x^2(\xi^1, \xi^2) &= \frac{|\xi^1|}{\xi^1} \rho_o(\xi^2) \; [e^{\alpha\theta_o|\xi^1|} \sin(\theta_o|\xi^1| + 40°) - \sin 40°] \\
x^3(\xi^1, \xi^2) &= Z_o\xi^2
\end{aligned}\right\} \tag{2.3.12}$$

where constants α, θ_o, Z_o and function $\rho_o(\xi^2)$ are defined by

$$\left.\begin{aligned}
\alpha &= \text{tg } 40° \\
\theta_o &= 48° \; 178 \\
Z_o &= 157
\end{aligned}\right\} \tag{2.3.13}$$

$$\rho_o(\xi^2) = 200 - 0.008233(Z_o)^2(\xi^2)^2 + 0.000029(Z_o)^3(\xi^2)^3 \tag{2.3.14}$$

3 - DEFORMATION OF A THIN SHELL

3.1. Basic hypotheses

The basic idea of Koiter's thin shell theories is to reduce the study of the deformation of a thin shell to the determination of the *displacement field* $\vec{u} = u_i\vec{a}^i$ of the particles of the middle surface.

In this way, Koiter uses the two following *basic hypotheses*

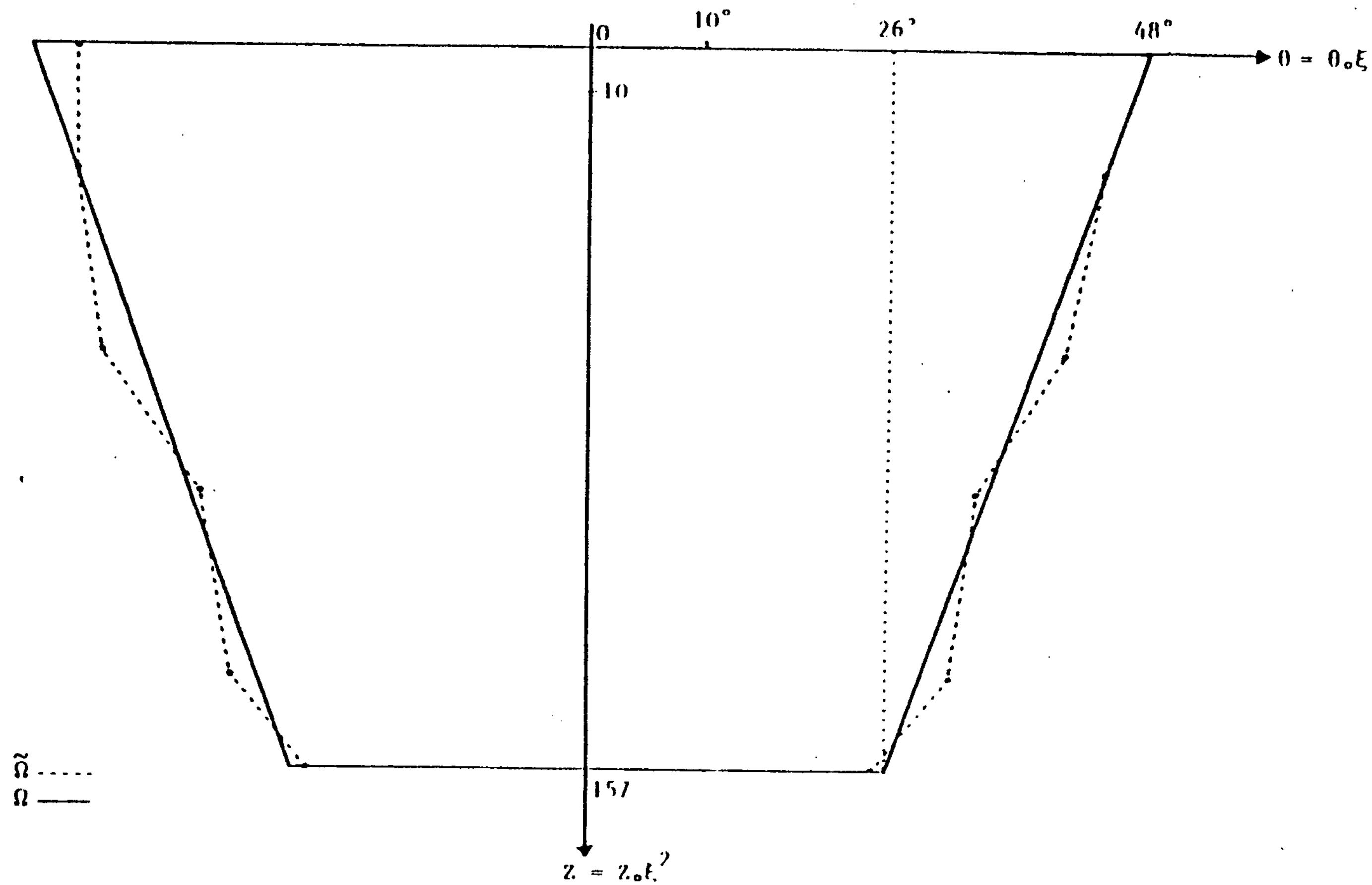


Figure 2.3.7 : Exact and approached reference domain $\tilde{\Omega}$ and Ω

(i) the normal to the undeformed middle surface, considered as a set of points of the shell, remains normal to the deformed middle surface ;

(ii) during the deformation, the stresses are approximatively plane and parallel to the tangent plane to the middle surface.

Such hypotheses allow us to obtain an approximation of the strain tensor γ^*_{ij} of the three-dimensional medium defined by

$$\gamma^*_{ij} = \frac{1}{2} (\bar{g}_{ij} - g_{ij}) \qquad (3.1.1)$$

where $\bar{g}_{ij}$ (respectively g_{ij}) is the metric tensor of the continuous medium in the deformed (respectively undeformed) configuration, for a same parameterization (ξ^1, ξ^2, ξ^3). The exponent * which appears in (3.1.1) indicates a difference between tensors in $\mathscr{E}^3$, on the one hand, and tensors defined on the middle surface, on the otherhand.

For instance, let us give a brief presentation of one possible modelization :

The assumptions of small displacements and small strains, added to that of a thin shell, allow us to obtain

$$\left.\begin{aligned} &\gamma^*_{\alpha\beta} = \frac{1}{2} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) - (\bar{b}_{\alpha\beta} - b_{\alpha\beta}) \xi^3 \quad , \\ &\gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = 0 \quad . \end{aligned}\right\} \qquad (3.1.2)$$

Moreover, for an *elastic homogeneous isotropic material* satisfying *HOOKE's law*, we can check that the second complementary hypothesis (ii) of plane stresses implies

$$\gamma^*_{33} = - \frac{\nu}{1-\nu} \gamma^{*\alpha}_{\ \alpha} \quad \text{with} \quad \gamma^{*\alpha}_{\ \beta} = g^{\alpha\lambda} \gamma^*_{\beta\lambda} \quad , \qquad (3.1.3)$$

where ν is *POISSON's coefficient* of the material. In order words, these relations show that the evaluation of the strain tensor of the shell $\mathscr{E}^*$ depends on the evaluation of the two following surface tensors :

(i) *the strain tensor of the middle surface*

$$\gamma_{\alpha\beta} = \frac{1}{2} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \quad , \qquad (3.1.4)$$

(ii) *the change of curvature tensor of the middle surface*

$$\bar{\rho}_{\alpha\beta} = \bar{b}_{\alpha\beta} - b_{\alpha\beta} \quad . \qquad (3.1.5)$$

After many simplifications, we can obtain an expression of tensors $\gamma_{\alpha\beta}$ and $\bar{\rho}_{\alpha\beta}$ as functions of the displacement vector $\vec{u}$ of the particles located upon the middle surface :

$$\gamma_{\alpha\beta}(\vec{u}) = \frac{1}{2} (u_{\beta|\alpha} + u_{\alpha|\beta}) - b_{\alpha\beta} u_3 \quad , \qquad (3.1.6)$$

$$\bar{\rho}_{\alpha\beta}(\vec{u}) = u_{3|\alpha\beta} - b^{\lambda}_{\alpha}b_{\lambda\beta}u_3 + b^{\lambda}_{\beta|\alpha}u_{\lambda} + b^{\lambda}_{\beta}u_{\lambda|\alpha} + b^{\lambda}_{\alpha}u_{\lambda|\beta} \quad , \tag{3.1.7}$$

where

$$u_{3|\alpha\beta} = u_{3,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta}u_{3,\lambda} \quad . \tag{3.1.8}$$

3.2. Total potential energy of the shell

In the previous section, we have developed an approximation of the strain tensor of the three-dimensional shell by assuming that the material is elastic, homogeneous and isotropic.

More generally, we can obtain an approximation of the total potential energy of the shell valid for more general elastic material (for instance, for orthotropic elastic shells). Thus, let us assume that the shell is

(i) *loaded* by a distribution of forces whose resultant is $\vec{p}$ on the middle surface S and whose resultant moment is $\vec{0}$ on S ;

(ii) *clamped* on the part Γ_o of its boundary $\Gamma = \partial\Omega$;

(iii) *loaded* on the complementary part $\Gamma_1 = \Gamma - \Gamma_o$ of its boundary by a distribution of forces whose resultant is $\vec{N}$ on Γ_1 and whose resultant moment is $\vec{M} = M^{\alpha}\vec{a}_{\alpha}$ on Γ_1.

Then, the corresponding *total potential energy* of the shell for an admissible displacement field $\vec{v}$ is given by

$$\left.\begin{aligned} P(\vec{v}) = \frac{1}{2}\int_{\Omega} [n^{\alpha\beta}(\vec{v})\gamma_{\alpha\beta}(\vec{v}) + m^{\alpha\beta}(\vec{v})\bar{\rho}_{\alpha\beta}(\vec{v})]\sqrt{a}\, d\xi^1 d\xi^2 \\ - \int_{\Omega} \vec{p}\,\vec{v}\sqrt{a}\, d\xi^1 d\xi^2 - \int_{\Gamma_1} [\vec{N}\,\vec{v} + M^{\alpha}\phi_{\alpha}(\vec{v})]\, d\gamma \end{aligned}\right\} \tag{3.2.1}$$

where

$(n^{\alpha\beta})$ = symmetrical tensor of stress resultants

$(m^{\alpha\beta})$ = symmetrical tensor of stress couples

$(\gamma_{\alpha\beta})$ = strain tensor on the middle surface

$(\bar{\rho}_{\alpha\beta})$ = tensor of changes of curvature

dS = $\sqrt{a}\, d\xi^1 d\xi^2$: area element on S .

It remains now to use the behaviour law in order to connect $n^{\alpha\beta}$ with $\gamma_{\alpha\beta}$ and $m^{\alpha\beta}$ with $\bar{\rho}_{\alpha\beta}$.

3.3. Complementary hypotheses

Subsequently, for simplicity, we assume the shell is elastic, homogeneous and isotropic, the deformations are small, the stresses are approximatively plane. So, we have the following relations

$$n^{\alpha\beta}(\vec{v}) = e\, E^{\alpha\beta\lambda\mu}\,\gamma_{\lambda\mu}(\vec{v}) \;,\; m^{\alpha\beta}(\vec{v}) = \frac{e^3}{12}\, E^{\alpha\beta\lambda\mu}\,\bar{\rho}_{\lambda\mu}(\vec{v}) \quad , \tag{3.3.1}$$

where

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} [a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu}] \quad . \tag{3.3.2}$$

When substituting (3.3.1) into (3.2.1), we find out that it just remains to define the functions

$$\vec{v} \to \gamma_{\alpha\beta}(\vec{v}) \quad \text{and} \quad \vec{v} \to \bar{\rho}_{\alpha\beta}(\vec{v}) \tag{3.3.3}$$

in order to completely determine the functional $P(\vec{v})$. In section 3.5, we will see four suitable possible choices (there are many others !).

3.4. Characterization of the solutions

Possible *solutions* $\vec{u}$ *minimize the total potential energy* $P(\vec{v})$ on an admissible displacement space $\vec{V}$. For sufficiently smooth data, these solutions $\vec{u}$ can also be characterized as solutions of

$$P'(\vec{u}).\vec{v} = 0 \quad , \quad \forall\vec{v} \in \vec{V} \quad , \tag{3.4.1}$$

where $P'(\vec{u})$ is the Fréchet derivative of P at point $\vec{u}$. Thanks to equations (3.3.1) and (3.3.2), relation (3.2.1) gives

$$\left.\begin{aligned} P'(\vec{u}).\vec{v} = \int_\Omega [n^{\alpha\beta}(\vec{u})(\gamma_{\alpha\beta})'(\vec{u})\vec{v} + m^{\alpha\beta}(\vec{u})(\bar{\rho}_{\alpha\beta})'(\vec{u})\vec{v} - \vec{p}\vec{v}] \sqrt{a}\, d\xi^1 d\xi^2 \\ - \int_{\Gamma_1} [\vec{N}\vec{v} + M^\alpha \phi_\alpha(\vec{v})]\, d\gamma \end{aligned}\right\} \tag{3.4.2}$$

Now, by expliciting the functions $\vec{u} \to \gamma_{\alpha\beta}(\vec{u})$ and $\vec{u} \to \bar{\rho}_{\alpha\beta}(\vec{u})$, we obtain four classical cases of shell equations.

3.5. Some examples of thin shell models ; corresponding existence results

Linear or nonlinear "general" shell equations

For the case of small deformations and small finite deflections, we consider the following expressions for $\gamma_{\alpha\beta}$ and $\bar{\rho}_{\alpha\beta}$ (non linear terms are underlined) :

$$\gamma_{\alpha\beta}(\vec{v}) = \theta_{\alpha\beta}(\vec{v}) + \frac{1}{2}\, \underline{a_{\alpha\beta}(\Omega(\vec{v}))^2} + \frac{1}{2}\, \underline{\phi_\alpha(\vec{v})\phi_\beta(\vec{v})} \tag{3.5.1}$$

$$\bar{\rho}_{\alpha\beta}(\vec{v}) = \frac{1}{2}\, [\phi_\alpha(\vec{v})|_\beta + \phi_\beta(\vec{v})|_\alpha + b_\alpha^\lambda(v_{\lambda|\beta} - b_{\lambda\beta}v_3) + b_\beta^\lambda(v_{\lambda|\alpha} - b_{\lambda\alpha}v_3)] \tag{3.5.2}$$

where we have set $\theta_{\alpha\beta}(\vec{v}) = 1/2\ (v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3$, $\phi_\alpha(\vec{v}) = v_{3|\alpha} + b_\alpha^\lambda v_\lambda$ and $\Omega(\vec{v}) = 1/2\ \epsilon^{\alpha\beta}\omega_{\alpha\beta}(\vec{v})$. These relations imply immediately

$$\begin{cases} (\gamma_{\alpha\beta})'(\vec{u})\vec{v} = \theta_{\alpha\beta}(\vec{v}) + \underline{a_{\alpha\beta}\Omega(\vec{u})\Omega(\vec{v})} + \underline{\phi_\alpha(\vec{u})\ \phi_\beta(\vec{v})} \\ (\bar{\rho}_{\alpha\beta})'(\vec{u})\vec{v} = \bar{\rho}_{\alpha\beta}(\vec{v}) \end{cases} .$$

Theorem 3.5.1 (BERNADOU and CIARLET [1976])

The linear general thin shell problem associated to the equations (3.5.1) *(linearized) and* (3.5.2), *has one and only one solution in the admissible displacement space* :

$$\vec{V} = \{\vec{v} \in (H^1(\Omega))^2 \times H^2(\Omega)\ ;\ \vec{v}|_{\Gamma_o} = \frac{\partial v_3}{\partial n}|_{\Gamma_o} = 0\}. \tag{3.5.3}$$

Proof : The proof takes essentially three steps :

*) *the square root of the strain energy*

$$I(\vec{v}) = \frac{1}{2}\int_\Omega \{n^{\alpha\beta}(\vec{v})\gamma_{\alpha\beta}(\vec{v}) + m^{\alpha\beta}(\vec{v})\bar{\rho}_{\alpha\beta}(\vec{v})\}\ \sqrt{a}\ d\xi^1 d\xi^2$$

is a norm on the space. $\vec{V}$. The main difficulty is to prove that $I(\vec{v}) = 0 \Rightarrow \vec{v} = \vec{0}$ *in the Sobolev space* $\vec{V}$. This is obtained by proving *first* that

$$\{I(\vec{v}) = 0 \quad , \quad \vec{v} \in (H^1(\Omega))^2 \times H^2(\Omega)\}$$

implies $\vec{v}$ is an infinitesimal rigid body motion. Then, by adding boundary conditions, this rigid body motion is reduced to $\vec{0}$.

**) *the energy norm* $\sqrt{I(\vec{v})}$ *is equivalent to the usual norm on the product space* $(H^1)^2 \times H^2$. This is completed by using arguments of lower weakly semi-continuity.

***) *then we apply the Lax-Milgram lemma* to the variational formulation (3.4.1).

□

Nonlinear case

The existence of solutions for the nonlinear *general* this shell problem seems to be *an open question*. We have investigated the *incremental methods* which are successful for some nonlinear elasticity problems (see BERNADOU-CIARLET and HU [1984]). Due to the absence of regularity results, these methods do not seem to be transposable even for some particular cases like clamped shell along all its boundary.

□

Linear or nonlinear "shallow" shell equations

By *shallow shell*, we intend a shell which has a weak curvature, in other words, a shell such that $b_{\alpha\beta}$ and $b_{\alpha\beta|\lambda}$ are very small when compared to the unity. These geometrical restrictions authorize to do some simplifications in equations (3.5.1) and (3.5.2). They can be respectively rewritten as follows :

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}\ (v_{\alpha|\beta}+v_{\beta|\alpha}) - b_{\alpha\beta}v_3 + \underline{\frac{1}{2}\ v_{3,\alpha}v_{3,\beta}} \tag{3.5.4}$$

$$\bar{\rho}_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta} \tag{3.5.5}$$

so that we immediately find

$$(\gamma_{\alpha\beta})'(\vec{u})\vec{v} = \theta_{\alpha\beta}(\vec{v}) + \underline{u_{3,\alpha}v_{3,\beta}}$$

$$(\bar{\rho}_{\alpha\beta})'(\vec{u})\vec{v} = \bar{\rho}_{\alpha\beta}(\vec{v}) \ .$$

Theorem 3.5.2 (BERNADOU and LALANNE [1985])

The linear shallow shell problem associated to equations (3.5.4) *(linearized) and* (3.5.5), *has one and only one solution in the admissible displacement* $\vec{V}$ *given by* (3.5.3) *as soon as the curvature of the middle surface of the shell is sufficiently small (i.e., "sufficiently shallow" shell).*

□

Theorem 3.5.3 (BERNADOU and ODEN [1981])

For "sufficiently shallow" shells and for "sufficiently small" tangential loads, the nonlinear shallow shell problem associated to the expressions (3.5.4) *and* (3.5.5) *has at least one solution. There is uniqueness when the normal load* p^3 *is sufficiently small.*

Proof : For shallow shells the nonlinearity concerns only u_3. Hence

i) we can solve the problem with respect to u_1 and u_2 (Lax-Milgram lemma) ;

ii) we substitute u_1 and u_2 into the third equation and we obtain a nonlinear equation $\mathcal{A}u_3 = f$;

iii) then, we show that operator $\mathcal{A}$ is coercive and pseudomonotone, so that $\mathcal{A}$ is surjective (LIONS [1969]).

□

PART II : APPROXIMATION OF THIN SHELLS BY FINITE ELEMENT METHODS

4 - APPROXIMATION BY CONFORMING FINITE ELEMENT METHODS

In this paragraph, we give an account of basic results obtained in the approximation of the general linear thin shell problems by using conforming finite element methods.

4.1. The discrete space $\vec{V}_h$

The definition of the middle surface S through the application $\vec{\phi}$ is very interesting with respect to the approximation by finite element methods. Indeed, according to (3.2.1), the problem is defined on *the plane reference domain* Ω instead to be defined on the middle surface S. And now, the geometry of the shell appears only *through variable coefficients*. More precisely, the variational formulation (3.4.1) can be written

$$\left.\begin{aligned} &\int_\Omega eE^{\alpha\beta\lambda\mu}\,[\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{u})\bar{\rho}_{\lambda\mu}(\vec{v})]\,\sqrt{a}\,d\xi^1 d\xi^2 \\ &\qquad = \int_\Omega \vec{p}\vec{v}\,\sqrt{a}\,d\xi^1 d\xi^2 - \int_{\Gamma_1} [\vec{N}\vec{v} + M^\alpha\phi_\alpha]\,d\gamma \quad . \end{aligned}\right\} \tag{4.1.1}$$

For simplicity, let us assume that $\vec{N} = \vec{0}$, $M^\alpha = 0$, so that (4.1.1) can be equivalently rewritten as

$$\int_\Omega {}^tU\,[A_{IJ}]\,V\,d\xi^1 d\xi^2 = \int_\Omega {}^tF\,V\,d\xi^1 d\xi^2 \tag{4.1.2}$$

where the column matrix V (and similarly for the matrix U) is given by

$${}^tV = [v_1\ v_{1,1}\ v_{1,2}\ v_2\ v_{2,1}\ v_{2,2}\ v_3\ v_{3,1}\ v_{3,2}\ v_{3,11}\ v_{3,12}\ v_{3,22}] \tag{4.1.3}$$

and where the square matrix $[A_{IJ}]_{12\times12}$ is only dependent of e, E, ν and of the first, second and third partial derivatives of the mapping $\vec{\phi}$.

In order to approximate problem (4.1.2), we have

i) *to construct a finite dimensional subspace* $\vec{V}_h$ of space $\vec{V}$. For this, we assume that Ω is a polygonal domain and we introduce finite element spaces

$$\left.\begin{aligned} &V_{h1} \subset V_1 = \{v \in H^1(\Omega)\ ,\ v|_{\Gamma_o} = 0\}\ , \\ &V_{h2} \subset V_2 = \{v \in H^2(\Omega)\ ,\ v|_{\Gamma_o} = \frac{\partial v}{\partial n}\Big|_{\Gamma_o} = 0\}\ , \\ &\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}\ . \end{aligned}\right\} \tag{4.1.4}$$

ii) to introduce a *numerical integration scheme*, i.e., for any element K of the triangulation $\mathcal{T}_h$:

$$\int_K \phi(\xi)\,d\xi \sim \sum_{\ell=1}^{L} \omega_{\ell,K}\,\phi(b_{\ell,K})\ . \tag{4.1.5}$$

4.2. The approximate problem

Thus, from (4.1.2), we obtain

$$\left.\begin{aligned} a_h(\vec{u}_h,\vec{v}_h) &= \sum_{K\in\mathscr{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \, ({}^tU_h \, [A_{IJ}] \, V_h) \, (b_{\ell,K}) \, , \\ f_h(\vec{v}_h) &= \sum_{K\in\mathscr{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \, ({}^tF \, V_h) \, (b_{\ell,K}) \, . \end{aligned}\right\} \qquad (4.2.1)$$

and the approximate problem can be stated as follows :

$$\textit{Find } \vec{u}_h \in \vec{V}_h \textit{ such that } a_h(\vec{u}_h,\vec{v}_h) = f_h(\vec{v}_h) \, , \ \forall \vec{v}_h \in \vec{V}_h \, . \qquad (4.2.2)$$

The mathematical analysis of such an approximation is complicated owing to the presence of variable coefficients and to the coupling between tangential components and normal component of the displacement. All proofs concerning such studies can be found in BERNADOU [1980] and BERNADOU-BOISSERIE [1982].

The Figure 4.2.1 gives some illustrations of these results in case of an approximation of high degree of precision.

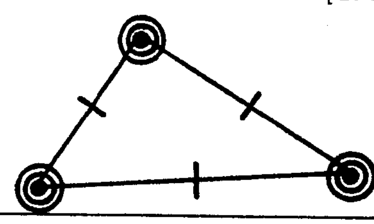
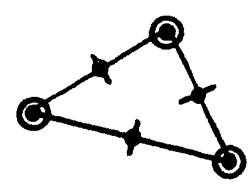
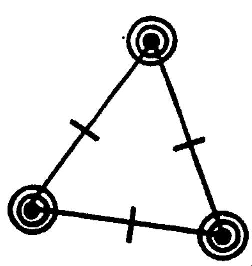
V_{h1} (u_1 and u_2) \ $V_{h2}(u_3)$	ARGYRIS-FRIED-SCHARPF [1968]	GANEV-DIMITROV [1980]
		
ARGYRIS 	$\Vert\vec{u}-\vec{u}_h\Vert = O(h^4)$ $\vec{u} \in (H^5(\Omega))^2 \times H^6(\Omega)$, $A_{IJ} \in W^{4,\infty}(\Omega)$ $p^i \in W^{4,q}(\Omega)$, $q \geq 2$ Integration scheme exact for polynomials of degree 8 (16 nodes)	$\Vert\vec{u}-\vec{u}_h\Vert = O(h^4)$ $\vec{u} \in (H^5(\Omega))^2 \times H^6(\Omega)$, $A_{IJ} \in W^{4,\infty}(\Omega)$ $p^i \in W^{4,q}(\Omega)$, $q \geq 2$ Integration scheme exact for polynomials of degree 6 (12 nodes)

Figure 4.2.1 : Examples of high degree of precision

The space V_{h2} is construct from ARGYRIS finite elements. For the space V_{h1}, we can take $V_{h1} = V_{h2}$, but this is expensive and useless. *Indeed, in this case, the mathematical analysis shows that the optimal choice to construct V_{h1} is to use the GANEV's element.* For the same asymptotic error estimate, for the same regularity on the data, that means less degrees of freedom and less integration nodes. This is illustrated by the example considered in section 4.3.

The Figure 4.2.2 gives analogous results for approximations of low degrees of precision using for instance the H.C.T.-elements.

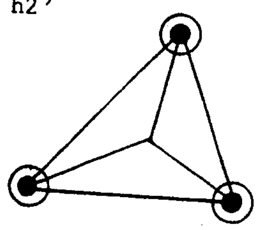
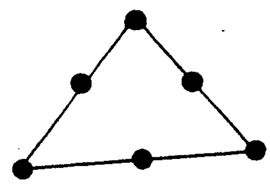
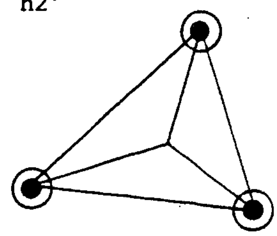
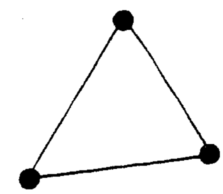
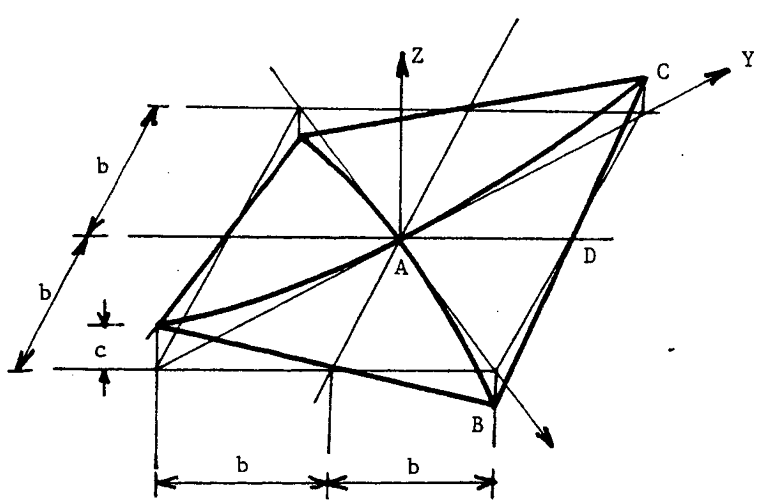
V_{h1}, V_{h2} } Complete H.C.T.	V_{h1} : Tr. type (2) V_{h2} : Complete H.C.T	V_{h1}, V_{h2} } Reduced H.C.T.	V_{h1} : Tr. type (1) V_{h2} : Reduced H.C.T
			
. $\Vert\vec{u}-\vec{u}_h\Vert = O(h^2)$. Scheme exact for P_4 on every sub-triangle (6 nodes) . $\vec{u} \in (H^3(\Omega))^2 \times H^4(\Omega)$	. $\Vert\vec{u}-\vec{u}_h\Vert = O(h^2)$. Scheme exact for P_2 on every sub-triangle (3 nodes) . $\vec{u} \in (H^3(\Omega))^2 \times H^4(\Omega)$	. $\Vert\vec{u}-\vec{u}_h\Vert = O(h)$. Scheme exact for P_4 on every sub-triangle (6 nodes) . $\vec{u} \in (H^3(\Omega))^3$	. $\Vert\vec{u}-\vec{u}_h\Vert = O(h)$. Scheme exact for P_2 on every sub-triangle (3 nodes) . $\vec{u} \in (H^2(\Omega))^2 \times H^3(\Omega)$

Figure 4.2.2 : Examples of error estimates, sufficient conditions on the quadrature schemes and regularity required for the solution

4.3. Test of ARGYRIS/ARGYRIS and ARGYRIS/GANEV methods on a clamped hyperbolic paraboloid roof under uniform pressure loading

As an illustration of previous considerations, we report hereunder some results obtained in the approximation of the classical bench-mark of a clamped hyperbolic paraboloid roof by the methods given in figure 4.2.1. This bench-mark has been considered by different authors CONNOR-BREBBIA [1967], BATOZ [1977] and ADINA [1983] ; it is described in figure 4.3.1.



Equation :

$Z = c(Y^2-X^2)/2a^2$

where

$a = 50$ cm

$c = 10$ cm

$h = 0.8$ cm

$E = 2.85.10^4$ kp/cm^2

$\nu = 0.4$

$q = 0.1$ kp/cm^2

Figure 4.3.1 : The hyperbolic paraboloid roof

The mesh in use is described in Figure 4.3.2 and takes into account the symmetry conditions. In Figure 4.3.2, we also indicate the number of degrees of freedom, the number of unknowns, the number of integration nodes and the computing time corresponding to both methods : ARGYRIS/ARGYRIS or ARGYRIS/GANEV. For more details see BERNADOU-HASSIM [to appear].

V_{h1} (u_1 and u_2)	ARGYRIS	GANEV
V_{h2} (u_3)	ARGYRIS	ARGYRIS
Number of d.o.f.	693	651
Number of unknowns	483	471
Number of integration nodes	576	432
Computing time (on BULL DPS8/70)		
. Rigidity matrix	693s	412s
. Second member	14s	9s
. Solution	153s	107s
Total	860s	528s

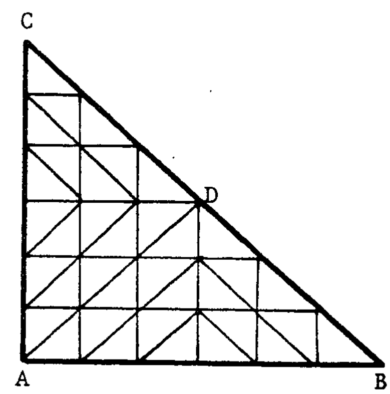


Figure 4.3.2 : The used mesh and some characteristic values of the implementation

Thus the ARGYRIS/GANEV method allows us to win about 38% of CPU time : 25% on the number of integration nodes plus about 20% on the obtention of elementary rigidity matrices (51 degrees of freedom for ARGYRIS/GANEV instead of 61 in case of ARGYRIS/ARGYRIS).

Let us add that the obtention of basis functions for the GANEV element has been realized by using the symbolic language MACSYMA [1983]. It is quite sure that the systematic use of such language for all the implementation could significantly improve the total computing time.

5 - APPROXIMATION BY FLAT PLATE ELEMENT METHODS

In this paragraph, we describe a very popular approximation method for the geometry of the shell which amounts to replace the given middle surface $\bar{S}$ by a *faceted middle surface* $\bar{S}_h$. Then, the energy of the shell is approached by a *sum of plate energies defined on each facet* of the surface $\bar{S}_h$ by using suitable finite element spaces. The most important difficulty is to propose good *compatibility conditions* from a facet to another in order that the approximated energy is sufficiently closed to the exact energy. Here, we briefly detail the method using CLOUGH-JOHNSON [1968] flat plate finite elements.

5.1. The approximate middle surface $\bar{S}_h$

Each component $\phi^i(\xi^1,\xi^2)$, $(\xi^1,\xi^2) \in \bar{\Omega}$, of the mapping $\vec{\phi}$ is replaced by its interpolant ϕ_h^i in the finite element space Φ_h associated to the P_1-triangle. So, the middle surface $\bar{S}$ is replaced by a continuous faceted surface $\bar{S}_h$ whose vertices are on $\bar{S}$ (see Figure 5.1.1). By using the mapping $\vec{\phi}_h = \phi_h^i \vec{e}_i$, we can define local basis $\vec{a}_{hi} \ldots$ exactly like in section 2.1.

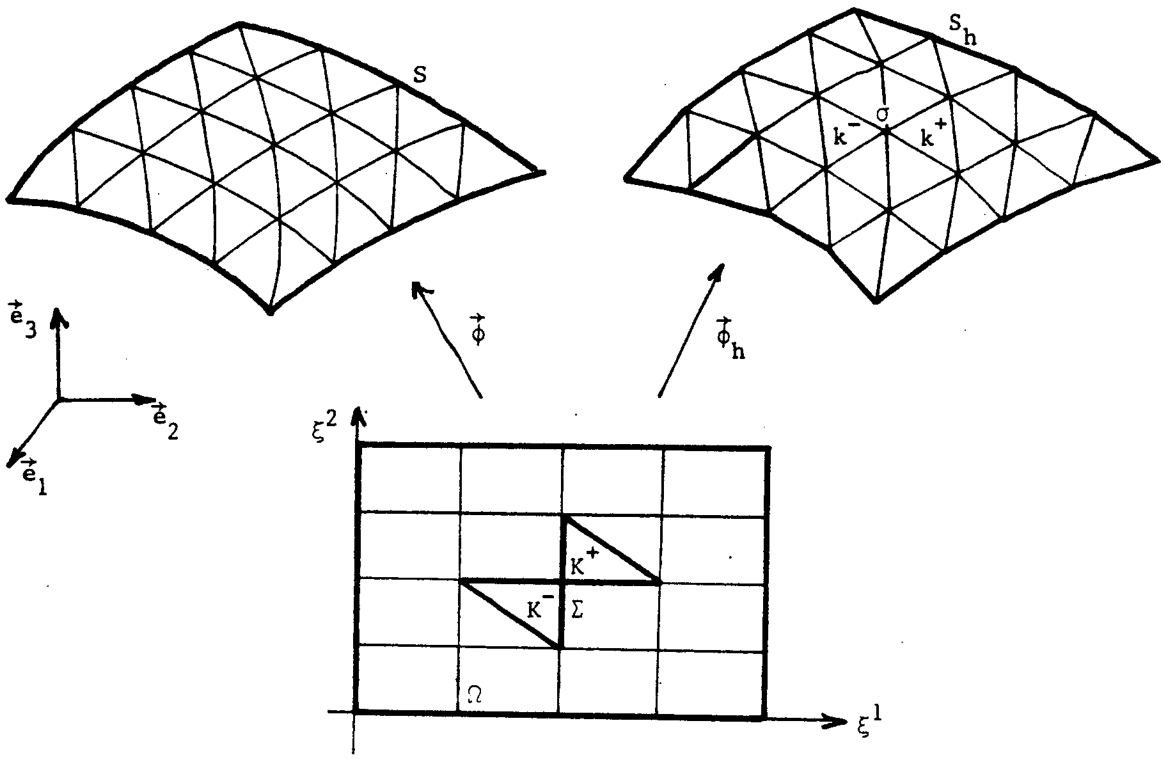


Figure 5.1.1 : The initial middle surface S and the faceted middle surface S_h

5.2. The discrete space $\vec{\tilde{V}}_h$ using compatibility conditions

Then, *on each facet*, we are able to define the corresponding potential energy. Next, *facet by facet*, we approach such an energy by a classical conforming finite element method for plate. For instance, in CLOUGH and JOHNSON method, on every triangular facet, the membrane displacements are approximated by a P_1-triangle meanwhile the deflection is approximated by a reduced HCT-triangle (see CLOUGH and TOCHER [1965]). At this stage, that amounts to introduce a discrete space

$$\vec{\tilde{X}}_h = \tilde{X}_{h1} \times \tilde{X}_{h1} \times \tilde{X}_{h2}$$

in which we approximate, facet by facet, the components $\tilde{v}_{h1}, \tilde{v}_{h2}$ and $\tilde{v}_{h3}$ of the displacement ($\vec{\tilde{v}}_h = \tilde{v}_{hi}\, \vec{a}_h^i$). These definitions of the spaces $\tilde{X}_{h1}$ and $\tilde{X}_{h2}$ involve immediately that the space $\vec{\tilde{X}}_h$ has 15 M_h degrees of freedom, where M_h denotes the number of triangles of the triangulation.

In order to get an approximate energy which is consistent (i.e. sufficiently closed) with the exact energy of the shell, we must introduce *constraints*, i.e., *compatibility conditions*, on the functions of the space $\vec{\tilde{X}}_h$: consider two facets k^+ and k^- of the approximate middle surface S_h which have a common vertex σ (see Figure 5.1.1). In the CLOUGH and JOHNSON approach, these compatibility conditions are :

i) *the displacement* $\vec{\vec{\vec{v}}}_h$ *is continuous* at the vertices σ of the surface S, or equivalently, at the vertices Σ of the triangulations $\mathcal{T}_h$, i.e.,

$$\vec{\vec{\vec{v}}}_h(\Sigma^+) = \vec{\vec{\vec{v}}}_h(\Sigma^-) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \; . \tag{5.2.1}$$

ii) *the tangential components* - with respect to the initial middle surface S - *of the rotation vector* $\vec{\vec{\vec{\omega}}}_h(\vec{\vec{\vec{v}}}_h)$ are continuous at the vertices σ of the surface S, or equivalently, at the vertices Σ of $\mathcal{T}_h$, i.e.,

$$\vec{\vec{\vec{\omega}}}_h(\Sigma^+).\vec{a}^{\alpha}(\Sigma) = \vec{\vec{\vec{\omega}}}_h(\Sigma^-).\vec{a}^{\alpha}(\Sigma) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \; . \tag{5.2.2}$$

Then the space $\vec{\vec{X}}_h$ is the set of functions $\vec{\vec{\vec{v}}}_h \in \vec{\vec{\vec{X}}}_h$ which satisfy the compatibility conditions (5.2.1) and (5.2.2). By adding suitable boundary conditions, we can define the subspace $\vec{\vec{V}}_h \subset \vec{\vec{X}}_h$ and prove the existence of a bijective correspondence F_h between the space $\vec{\vec{V}}_h$ and the space $\vec{V}_h$ introduced in the last example given in Figure 4.2.2.

By using such a method, CLOUGH and JOHNSON have obtained good numerical results, specially for cylindrical shells.

5.3. The discrete problem (CLOUGH-JOHNSON method)

The energy of the shell is now approximated by the sum of plate energies defined upon each facet of the approximate middle surface S_h. So doing, we define the bilinear form $\tilde{a}_h(.,.)$:

$$\left.\begin{aligned} \tilde{a}_h(\vec{\vec{u}}_h,\vec{\vec{v}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2} \{(1-\nu)\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\gamma}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\gamma}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h) \\ + \frac{e^2}{12}\,[(1-\nu)\tilde{\rho}^{\ \alpha}_{h\beta}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\alpha}(\vec{\vec{v}}_h) + \nu\tilde{\rho}^{\ \alpha}_{h\alpha}(\vec{\vec{u}}_h)\tilde{\rho}^{\ \beta}_{h\beta}(\vec{\vec{v}}_h)]\}\sqrt{a_h}\,d\xi^1 d\xi^2 \quad , \quad \forall\vec{\vec{u}}_h,\vec{\vec{v}}_h \in \vec{\vec{V}}_h \end{aligned}\right\} \tag{5.3.1}$$

where the components of the strain and change of curvature tensors are given by

$$\tilde{\gamma}_{h\alpha\beta}(\vec{\vec{v}}_h) = \frac{1}{2}\,(\tilde{v}_{h\beta,\alpha} + \tilde{v}_{h\alpha,\beta}) \quad ; \quad \tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h) = \tilde{v}_{h3,\alpha\beta} \; . \tag{5.3.2}$$

Likewise, we approximate the potential energy of external loads by the expressions

$$\tilde{f}_h(\vec{\vec{v}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \vec{p}.\vec{\vec{v}}_h \sqrt{a_h}\,d\xi^1 d\xi^2 \quad , \quad \forall\vec{\vec{v}}_h \in \vec{\vec{V}}_h \; . \tag{5.3.3}$$

Next, we introduce a supplementary approximation in order to take into account *numerical integration* used to compute the integrals $\int_K(\ldots)$ which appear in relations (5.3.1) and (5.3.3). This procedure amounts to approximate the forms $\tilde{a}_h(.,.)$ and $\tilde{f}_h(.)$ by new forms $\tilde{a}^*_h(.,.)$ and $\tilde{f}^*_h(.)$. So, we obtain the following definition of the *flat facet discrete problem* :

Problem 5.3.1 : Find $\vec{\vec{u}}^*_h \in \vec{\vec{V}}_h$ *such that*

$$\tilde{a}^*_h(\vec{\vec{u}}^*_h,\vec{\vec{v}}_h) = \tilde{f}^*_h(\vec{\vec{v}}_h) \quad , \quad \forall\vec{\vec{v}}_h \in \vec{\vec{V}}_h \; . \tag{5.3.4}$$

By using the bijection F_h between spaces $\vec{\vec{V}}_h$ and $\vec{V}_h$, we can define the approximate forms $b_h(\vec{u}_h,\vec{v}_h) = \tilde{a}^*_h(\vec{\vec{u}}_h,\vec{\vec{v}}_h)$ and $g_h(\vec{v}_h) = \tilde{\vec{f}}^*_h(\vec{v}_h)$, and then, we obtain the equivalent formulation :

Problem 5.3.2 : Find $\vec{u}^*_h \in \vec{V}_h$ *such that*

$$b_h(\vec{u}^*_h,\vec{v}_h) = g_h(\vec{v}_h) \quad , \quad \forall\vec{v}_h \in \vec{V}_h \ . \tag{5.3.5}$$

5.4. Study of consistency errors

As a basis of convergence properties, we have to study the consistency errors.

Abstract error estimate :

Theorem 5.4.1 : Let us consider a family of discrete problems (5.3.5) *for which the bilinear forms* $b_h(.,.)$ *are* $\vec{V}_h$*-elliptic, uniformly with respect to* h. *Then, there exists a constant* C, *independent of* h, *such that*

$$\left.\begin{aligned} \|\vec{u}-\vec{u}_h\| \le C\Big(&\inf_{\vec{v}_h\in\vec{V}_h} \Big(\|\vec{u}-\vec{v}_h\| + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|}\Big) \\ &+ \sup_{\vec{w}_h\in\vec{V}_h} \frac{|f(\vec{w}_h)-g_h(\vec{w}_h)|}{\|\vec{w}_h\|}\Big) \end{aligned}\right\} \tag{5.4.1}$$

where $\vec{u}$ *(resp.* $\vec{u}^*_h$*) denotes the solution of the continuous problem* (4.1.1) *(resp. of the discrete problem* (5.3.5)).

□

We have now to study the two consistency terms $|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|$ and $|f(\vec{v}_h)-g_h(\vec{w}_h)|$. Let us detail the results for the first one.

Estimate of the consistency error $|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|$: It amounts to estimate the quantities $|\gamma^\alpha_\beta(\vec{v}_h)-\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{v}}_h)|_{0,K}$ and $|\bar{\rho}^\alpha_\beta(\vec{v}_h)-\tilde{\rho}^{\ \alpha}_{h\beta}(\vec{\vec{v}}_h)|_{0,K}$. The obtention of these estimates, particularly the one concerning the change of curvature tensor, constitutes the *main difficulty of this study*. We recall here the main lines of the results ; all the details of the proofs can be found in BERNADOU-DUCATEL and TROUVE [1987].

Theorem 5.4.2 : There exists a constant C, *independent of* h, *such that, for any* $\vec{\vec{v}}_h \in \vec{\vec{X}}_h$ *and* $\vec{v}_h \in \vec{X}_h$ *in correspondence through the bijection* F_h *between spaces* $\vec{\vec{V}}_h$ *and* $\vec{V}_h$, *we have*

$$|\gamma^\alpha_\beta(\vec{v}_h)-\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\vec{v}}_h)|_{0,K} \le Ch(\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{1,K})^{1/2} \ . \tag{5.4.2}$$

□

Concerning the consistency between change of curvature tensors, we prove in the next theorem (*main result of this paper*) that the result is different and much more complicated :

Theorem 5.4.3 : There exists constants $c^{\ell}_{j\alpha\beta}$, $c^{\ell\lambda}_{j\alpha\beta}$, $c^{\epsilon\eta}_{j\alpha\beta}$, *independent of* h, *such that, for any* $\vec{\vec{v}}_h \in \vec{\vec{X}}_h$ *and* $\vec{v}_h \in \vec{X}_h$ *in correspondence through the bijection* F_h *between spaces* $\vec{\vec{V}}_h$ *and* $\vec{V}_h$, *we have*

$$\left.\begin{aligned}\tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)\big|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\epsilon\mu}(\xi)\sum_{i=1}^{3} A_\lambda^{\epsilon\mu}(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta} \\ + (\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}] + 0(h)\sum_{k=1}^{3}[c^\ell_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k) + c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda} + c^{\epsilon\eta}_{j\alpha\beta}v_{h3,\epsilon\eta}(\bar{\xi}_k)]\end{aligned}\right\} \quad (5.4.3)$$

where K_j denotes any subtriangle of the reduced HCT-triangle, where $p^1_{j,i,i\pm1}$ denote the basis functions of the reduced HCT-triangle, where $A^{\epsilon\mu}_\beta(\xi) = (1/2)\sum_{i=1}^3 (\xi^\epsilon_i-\xi^\epsilon)(\xi^\mu_i-\xi^\mu)\,\partial\lambda_i/\partial\xi^\beta$ and where the λ_i are the barycentric coordinates of the considered triangle.

□

To obtain an estimate of type (5.4.2) from finite expansion (5.4.3), we have to prove that the term

$$\left.\begin{aligned}c_{\alpha\beta}(\xi) = a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\epsilon\mu}(\xi)\sum_{i=1}^{3} A_\lambda^{\epsilon\mu}(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta} \\ + (\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}]\end{aligned}\right\} \quad (5.4.4)$$

is in $0(h)$. Unfortunately the counterexample of a right circular cylindrical shell shows that $c_{\alpha\beta}(\xi)$ is only in $0(1)$ with respect to h.

From this ascertaining, we have explored two directions :

i) *the use of the "modified" change of curvature tensor* : such a modification does not improve the situation !

ii) *the case of "quasi-shallow" shells* : in this case, we have been able to prove a "pseudo-convergence" result for "sufficiently" shallow shells. By "pseudo-convergence", we mean that the estimate (5.4.1) leads to

$$\|\vec{u}-\vec{u}^*_h\| \leq C[h((\sum_{\alpha=1}^{2}\|u_\alpha\|^2_{2,\Omega}+\|u_3\|^2_{3,\Omega})^{1/2} + \|\vec{p}\|_{1,q,\Omega}) + \epsilon\|\vec{u}\|] \quad (5.4.5)$$

where ϵ denotes the maximum of the curvature of the shell.

The estimate (5.4.5) can be regarded only as a "pseudo-convergence" result because this result depends on two parameters : h which can be arbitrary small ; ϵ which characterizes the shallowness of the shell so that, for a given shell, this parameter is small but fixed.

Let us mention that this result corroborates numerical observations made by GALLAGHER [1976], IRONS and AHMAD [1980], ZIENKIEWICZ [1977]. These Authors observe a deterioration of numerical results, particularly in the case of preponderant flexural terms : in this case, the term (5.4.4) is not negligible.

Besides the above results, one of the main interest of the estimate (5.4.3) is to allow to propose a new flat facet approximation which is *unconditionnally convergent for arbitrary thin shells*. This is the purpose of the next paragraph.

5.5. A method convergent for arbitrary thin shell

All the difficulties that we have met in the study of the convergence of the previous method are coming from the presence of the term $c_{\alpha\beta}$ (see (5.4.4)) in the expression (5.4.3). But, from relation (5.4.2), we have

$$\gamma_{\nu\eta}(\vec{v}_h) = \tilde{\gamma}_{h\nu\eta}(\vec{\vec{v}}_h) + O(h)$$

and so, we can rewrite relation (5.4.3) as

$$\tilde{\rho}^{*}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + O(h) \tag{5.5.1}$$

where we have set

$$\left.\begin{aligned} \tilde{\rho}^{*}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} &= \tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)|_{K_j} - \\ &- a^{\nu\lambda}(\xi)\tilde{\gamma}_{h\nu\eta}(\vec{\vec{v}}_h)b_{\epsilon\mu}(\xi) \sum_{i=1}^{3} A^{\epsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\alpha\beta} \\ &\qquad + (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\alpha\beta}] \;. \end{aligned}\right\} \tag{5.5.2}$$

If we substitute $\tilde{\rho}^{*}_{h\alpha\beta}$ to $\tilde{\rho}_{h\alpha\beta}$, we then obtain from (5.5.1) the searched and decisive estimate of type (5.4.2).

Let us notice that :

i) the implementation of relation (5.5.2) does not present any new difficulties ;

ii) the second term of the right hand member of relation (5.5.2) can be regarded as a perturbation of the flexural plate energy on each plane facet ;

iii) the introduction of a perturbation in the flexural energy does not disturb the "well posed" character of the discrete problem.

So doing, we define the new discrete problem :

Problem 5.5.1 : Find $\vec{u}^{**}_h \in \vec{V}_h$ *such that*

$$\bar{b}_h(\vec{u}^{**}_h,\vec{v}_h) = g_h(\vec{v}_h) \quad , \quad \forall\vec{v}_h \in \vec{V}_h \;, \tag{5.5.3}$$

where g_h is defined as in Problem 5.3.2 and where $\bar{b}_h(.,.)$ is deduced (by using the bijection F_h) from the bilinear form $\tilde{a}^{*}_h(.,.)$ (see (5.3.4)) by substituting to $\tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h)$ the expression $\tilde{\rho}^{*}_{h\alpha\beta}(\vec{\vec{v}}_h)$ defined in (5.5.2).

□

Similarly to the previous analysis, we can deduce, from estimates (5.4.2) and (5.5.1), the following convergence theorem which is available for any general shells :

Theorem 5.5.1 : There exists a constant C, *independent of* h, *such that*

$$\|\vec{u}-\vec{u}_h^{**}\| \le Ch\{[\sum_{\alpha=1}^{2} \|u_\alpha\|^2_{2,\Omega} + \|u_3\|^2_{3,\Omega}]^{1/2} + \|\vec{p}\|_{1,q,\Omega}\}$$

where $\vec{u}_h^{**} \in \vec{V}_h$ *(resp.* $\vec{u} \in \vec{V}$*) is the solution of the discrete problem* (5.5.3) *(resp. of the continuous problem* (4.1.1)*).*

□

6 - MIXED FINITE ELEMENT METHODS

By using conforming finite element methods, we have obtained very good results with respect to the numerical analysis of the problem and to the numerical experiments, particularly in real-life situations. But such conforming methods present intrinsically two difficulties :

i) the main unknown is the displacement field $\vec{u}$ when the *main information required by engineers is the distribution of the stress field*. That means we need to derive the stresses from the displacement and so we get a distribution which is not really good ; in particular, stresses are discontinuous from an element to another ;

ii) *the use of finite elements of class* $\mathscr{C}^1$ which are not easy to implement.

The mixed finite element methods allow to circumvent these two difficulties but the major drawback is the computation of the discrete solution which generally requires some specialized duality techniques.

In this brief presentation we will just set *the new variational formulation* well adapted to this kind of approximation and then, we will define the corresponding discrete problem. For more details, see BERNADOU-BOUZIANE and THOMAS [to appear] and BOUZIANE [1984].

6.1. Mixed formulation of HERMANN-MIYOSHI type (continuous problem)

With the notations introduced in paragraph 3, this mixed formulation can be stated as follows (for simplicity, we assume the shell is clamped along its boundary) :

$$\left.\begin{array}{l} \textit{Find } n^{\lambda\mu} \in M \ , \ m^{\lambda\mu} \in M^* \text{ and } \vec{u} \in (H_o^1(\Omega))^3 \textit{ such that} \\ \int_\Omega \frac{1}{e} G_{\alpha\beta\lambda\mu}\, n^{\lambda\mu}\, \eta^{\alpha\beta} \sqrt{a}\, d\xi^1 d\xi^2 + \int_\Omega \frac{12}{e^3} G_{\alpha\beta\lambda\mu}\, m^{\lambda\mu}\, \tau^{\alpha\beta} \sqrt{a}\, d\xi^1 d\xi^2 \\ \qquad = \int_\Omega \{\gamma_{\alpha\beta}(\vec{u})\, \eta^{\alpha\beta} + f_{\alpha\beta}(\underline{u})\, \tau^{\alpha\beta} - u_{3|\alpha}\, \tau^{\alpha\beta}|_\beta\} \sqrt{a}\, d\xi^1 d\xi^2 \\ \text{for all } \underset{\approx}{\eta} \in M \ , \ \underset{\approx}{\tau} \in M^* \end{array}\right\} \quad (6.1.1)$$

and

$$\left.\begin{array}{l} \textit{Find } n^{\alpha\beta} \in M \text{ and } m^{\alpha\beta} \in M^* \textit{ such that} \\ \int_\Omega \{\gamma_{\alpha\beta}(\vec{v})\, n^{\alpha\beta} + f_{\alpha\beta}(\underline{v})\, m^{\alpha\beta} - v_{3|\alpha}\, m^{\alpha\beta}|_\beta\} \sqrt{a}\, d\xi^1 d\xi^2 \\ \qquad = \int_\Omega \vec{p}\vec{v} \sqrt{a}\, d\xi^1 d\xi^2 \ , \ \forall \vec{v} \in (H_o^1(\Omega))^3 \end{array}\right\} \quad (6.1.2)$$

where $G_{\lambda\mu\alpha\beta}$ is the inverse of $E^{\alpha\beta\lambda\mu}$ defined by (3.3.2), i.e.,

$$G_{\lambda\mu\alpha\beta} = \frac{1+\nu}{2E} \{a_{\alpha\lambda}a_{\beta\mu} + a_{\alpha\mu}a_{\beta\lambda} - \frac{2\nu}{1+\nu} a_{\alpha\beta}a_{\lambda\mu}\}$$

and

$$\underline{v} = (v_1,v_2) \quad , \quad f_{\alpha\beta}(\underline{v}) = \rho_{\alpha\beta}(\vec{v}) - v_{3|\alpha\beta}$$

$$M = \{\underset{\approx}{\tau} = (\tau^{\alpha\beta}) | \tau^{\alpha\beta} \in L^2(\Omega) \quad \text{and} \quad \tau^{12} = \tau^{21}\}$$

$$M^* = \{\underset{\approx}{\tau} = (\tau^{\alpha\beta}) | \tau^{\alpha\beta} \in H^1(\Omega) \quad \text{and} \quad \tau^{12} = \tau^{21}\} .$$

Note that the equation (6.1.1) is obtained by applying the Green formula to the first member of relation (4.1.1), whereas equation (6.1.2) is another form of the equation (4.1.1). Also note that we use here the alternative definition of the change of curvature tensor, i.e. (see KOITER [1966]) :

$$\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} - \frac{1}{2} (b^\nu_\alpha \gamma_{\nu\beta} + b^\nu_\beta \gamma_{\nu\alpha})$$

6.2. Mixed formulation of HERMANN-MIYOSHI type (discrete problem)

From now on, we assume that the domain Ω is of polygonal type and we consider a family of regular triangulations $\mathcal{T}_h$. Then, we introduce three different finite element spaces X_h, X_{ho} and Y_h in which we will approximate the components of the unknowns of the problem, i.e., $\vec{u} = u_i\vec{a}^i$, $\underset{\approx}{n} = (n^{\alpha\beta})$ and $\underset{\approx}{m} = (m^{\alpha\beta})$. More precisely, by denoting K the generic triangle of $\mathcal{T}_h$, we define

$$X_h = \{v_h \in \mathcal{C}^o(\bar{\Omega}) \quad ; \quad v_{h|K} \in P_{K1} \quad , \quad \forall K \in \mathcal{T}_h\} ,$$

$$X_{ho} = \{v_h \in X_h \quad ; \quad v_h = 0 \text{ on } \Gamma\} ,$$

$$Y_h = \{v_h \quad ; \quad v_{h|K} \in P_{K2} \quad , \quad \forall K \in \mathcal{T}_h\} .$$

Of course,

$$X_h \subset H^1(\Omega) \quad ; \quad X_{ho} \subset H^1_o(\Omega) \quad \text{and} \quad Y_h \subset L^2(\Omega) .$$

So, the *discrete problem* is defined as follows :

Find $\vec{u}_h \in (X_{ho})^3$, $\underset{\approx}{n}_h \in (Y_h)^4_S$, $\underset{\approx}{m}_h \in (X_h)^4_S$ *such that*

$$\left.\begin{aligned}
&\int_\Omega \frac{1}{e} G_{\alpha\beta\lambda\mu} n^{\lambda\mu}_h \eta^{\alpha\beta}_h \sqrt{a}\, d\xi^1 d\xi^2 + \int_\Omega \frac{12}{e^3} G_{\alpha\beta\lambda\mu} m^{\lambda\mu}_h \tau^{\alpha\beta}_h \sqrt{a}\, d\xi^1 d\xi^2 \\
&\qquad = \int_\Omega \{\gamma_{\alpha\beta}(\vec{u}_h) \eta^{\alpha\beta}_h + f_{\alpha\beta}(\underline{u}_h) \tau^{\alpha\beta}_h - u_{3h|\alpha} \tau^{\alpha\beta}_h|_\beta\} \sqrt{a}\, d\xi^1 d\xi^2 \\
&\text{for all } \underset{\approx}{\eta}_h = (\eta^{\alpha\beta}_h) \in (Y_h)^4_S \quad , \quad \underset{\approx}{\tau}_h = (\tau^{\alpha\beta}_h) \in (X_h)^4_S
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\int_\Omega \{\gamma_{\alpha\beta}(\vec{v}_h) n^{\alpha\beta}_h + f_{\alpha\beta}(\underline{v}_h) m^{\alpha\beta}_h - v_{3h|\alpha} m^{\alpha\beta}_h|_\beta\} \sqrt{a}\, d\xi^1 d\xi^2 \\
&\qquad = \int_\Omega \vec{p}\vec{v}_h \sqrt{a}\, d\xi^1 d\xi^2 \quad , \quad \forall \vec{v}_h \in (X_{ho})^3
\end{aligned}\right\}$$

where $(\ )^4_S$ denotes a quadruplet whose second and third components are equal.

To conclude, let us mention some other works related to the numerical analysis of dual mixed and hybrid finite element methods applied to thin shell problems : DESTUYNDER and LUTOBORSKI [1982], MIYOSHI [1973] for mixed formulations, KIKUCHI and ANDO [1974], STEPHAN and WEISGERBER [1978] for hybrid formulations.

7 - APPROXIMATION BY D.K.T. METHODS

The D.K.T. (Discrete Kirchhoff Triangle) methods are now popular when approximating linear or nonlinear *plate problems*. These methods are based on MINDLIN plate models which take into account the effects of transverse shear deformations and which are well adapted to moderately thick plates. Unfortunately, a direct discretization of these MINDLIN models by conforming finite element methods leads to numerical instabilities (locking phenomenon) as soon as the plate thickness becomes too small. A first way to overcome such instabilities is to use reduced integration technics ; a second way is to use *D.K.T. methods whose basic ideas are* :

i) when computing the strain energy, to *neglect the part due to transverse shear deformations* (generally small when compared to membrane and bending parts) ;

ii) *on the discrete model, to introduce some constraints* like

* some KIRCHHOFF-LOVE conditions (the normal to the middle surface remain normals after deformation) at the mesh nodes ;

** some tangential KIRCHHOFF-LOVE conditions at some nodes located on the element interfaces ;

*** some other constraints without a clear mechanical meaning, but insuring a good definition of the discrete problem.

In particular, these D.K.T. methods allow to compute plates whose thickness is varying from thin to moderately thick by using *the same discrete model*, without the above mentionned numerical instabilities.

In *case of plates*, these methods have been extensively studied by KIKUCHI [1981] with respect to the numerical analysis and by many Authors with respect to numerical experiments ; among them, we refer to BATOZ-BATHE and HO [1980] and to the references of this paper.

In case of shell approximations, the original idea of D.K.T. method appears in a paper from WEMPNER, ODEN and KROSS [1971]. Next, many numerical experiments were done for different kinds of shells : for a summary of contributions in this way,, we refer to BATOZ, BEN-TAHAR and DHATT [1982] and to the bibliography of this paper.

Hereunder, we give a short description of the formulation of a D.K.T. method in the context of the KOITER's formulation studied in paragraph 2, and we refer to BERNADOU and MATO EIROA [1987] for more details.

7.1. A set of thin shell equations including the effect of transverse shear deformations

In addition to the usual displacement field $\vec{u}$ of the particles located on the middle surface, we need to take the components β_α of the rotation of the normal $\vec{a}_3$ as additional unknowns. Then, the corresponding variational formulation can be written as

$$\left.\begin{aligned}
&\textit{Find } (\vec{u},\underline{\beta}) \in V^5 \quad , \quad \underline{\beta} = \beta_\alpha \vec{a}^\alpha \quad , \quad \textit{such that} \\
&\int_\Omega eE^{\alpha\beta\lambda\mu} \{\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12}\,\chi_{\alpha\beta}(\vec{v},\underline{\beta})\chi_{\lambda\mu}(\vec{v},\underline{\delta})\}\,\sqrt{a}\;d\xi^1 d\xi^2 \\
&\qquad + \int_\Omega \frac{Ea^{\alpha\beta}}{2(1+\nu)}\,\{\phi_\alpha(\vec{u}) + \beta_\alpha\}\,\{\phi_\beta(\vec{v}) + \delta_\beta\}\,\sqrt{a}\;d\xi^1 d\xi^2 \\
&\qquad = \int_\Omega \vec{p}.\vec{v}\,\sqrt{a}\;d\xi^1 d\xi^2 + \int_{\Gamma_1} (\vec{N}.\vec{v} - M^\alpha \delta_\alpha)\,d\gamma \quad , \quad \forall(\vec{v},\underline{\delta}) \in V^5 \quad ,
\end{aligned}\right\} \tag{7.1.1}$$

where

$$\left.\begin{aligned}
&V = \{v \mid v \in H^1(\Omega) \; ; \; v_{|\Gamma_0} = 0\} \quad , \\
&\chi_{\alpha\beta}(\vec{u},\underline{\beta}) = \frac{1}{2}\,[(\beta_{\alpha|\beta}+\beta_{\beta|\alpha}) - b^\lambda_\alpha(u_{\lambda|\beta} - b_{\lambda\beta}u_3) - b^\lambda_\beta(u_{\lambda|\alpha} - b_{\lambda\alpha}u_3)] \\
&\phi_\alpha(\vec{u}) = u_{3|\alpha} + b^\lambda_\alpha u_\lambda \; .
\end{aligned}\right\} \tag{7.1.2}$$

Note that this formulation gives back the usual KOITER's formulation when during the deformation the normal remains normal, i.e., when $\beta_\alpha + \phi_\alpha = 0$ or, equivalently, when $\beta_\alpha = -(u_{3|\alpha}+b^\lambda_\alpha u_\lambda)$.

7.2. The finite element spaces used to approximate $\vec{u}$ and $\underline{\beta}$

The unknowns $\vec{u}$ and $\underline{\beta}$ are approximated in finite dimensional spaces V_{h1} or V_{h2} constructed by using the elements mentionned in Figure 7.2.1. Of course, other choices of triangles or quadrilaterals are possible : see for instance BATOZ, BEN-TAHAR and DHATT [1982].

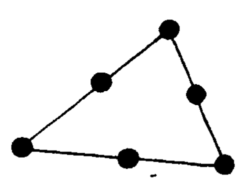
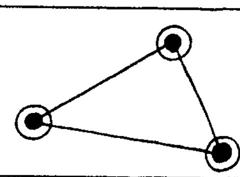
Unknowns	Finite element type	Schematization of the element	Corresponding finite element space
u_1 , u_2 , β_1 , β_2 ,	P_2-Lagrange		V_{h1}
u_3	P_3'-Hermite		V_{h2}

Figure 7.2.1 : Finite elements in use for D.K.T. approximation

Constraints imposed to the degrees of freedom

For a given triangle, there are 33 degrees of freedom. By imposing 12 constraints, we are going to reduce this number up to 21 effective degrees of freedom, i.e., with obvious notations : $u_{\alpha h}(a_i)$, $u_{\alpha h}(b_i)$, $u_{3h}(a_i)$, $\beta_{\alpha h}(a_i)$, $\alpha=1,2$, $i=1,2,3$, where a_i, b_i denote respectively the vertices and the midsides of the triangle. These 12 constraints are (see Figure 7.2.2) :

i) *The KIRCHHOFF-LOVE condition is verified at the vertices* a_i :

$$\beta_{\alpha h}(a_i) = - u_{3h,\alpha}(a_i) - b^{\lambda}_{\alpha}(a_i)u_{\lambda h}(a_i) \quad ; \tag{7.2.1}$$

ii) *The KIRCHHOFF-LOVE condition is "tangentially" satisfied at midsides* b_i :
One can check that $\vec{t}_i = t^{\alpha}_i \vec{a}_{\alpha}$ with $t^{\alpha}_i = \xi^{\alpha}_{i+1} - \xi^{\alpha}_{i-1}$ so that the constraints are

$$t^{\nu}_i \beta_{\nu h}(b_i) = - Du_{3h}(b_i).\overrightarrow{a_{i-1}a_{i+1}} - t^{\nu}_i b^{\lambda}_{\nu}(b_i)\, u_{\lambda h}(b_i) \quad . \tag{7.2.2}$$

iii) *The "normal" component of vector* $\underline{\beta}_h(b_i)$ *is equal to the mid-sum of normal components at the adjacent vertices* a_{i-1} *and* a_{i+1} *, i.e.,*

$$(\underline{\beta}_h.\vec{n}_i)(b_i) = \frac{1}{2}\,\{(\underline{\beta}_h.\vec{n}_i)(a_{i+1}) + (\underline{\beta}_h.\vec{n}_i)(a_{i-1})\}$$

or, equivalently,

$$(\epsilon^{\alpha\lambda} t_{\alpha i}\beta_{\lambda h})(b_i) = \frac{1}{2}\,\{(\epsilon^{\alpha\lambda} t_{\alpha i}\beta_{\lambda h})(a_{i+1}) + (\epsilon^{\alpha\lambda} t_{\alpha i}\beta_{\lambda h})(a_{i-1})\}$$

with

$$\epsilon^{\alpha\lambda} = \frac{1}{\sqrt{a}}\, e^{\alpha\lambda} \quad \text{and} \quad e^{11} = e^{22} = 0 \quad , \quad e^{12} = -e^{21} = 1 \ .$$

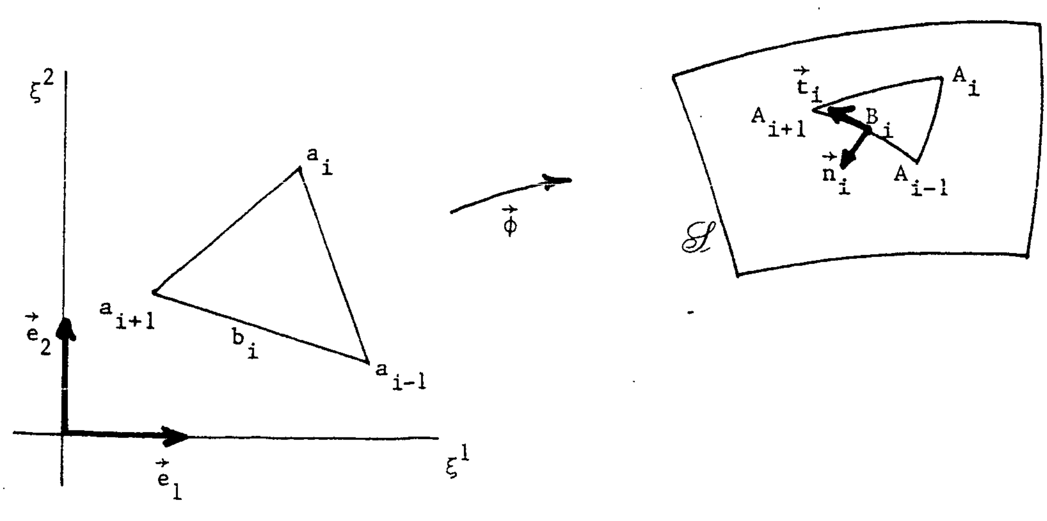


Figure 7.2.2 : Corresponding triangles

7.3. The D.K.T. approximated problem : *Find* $(\vec{u}_h, \underline{\beta}_h) \in \tilde{\tilde{V}}_h$ *such that*

$$\left.\begin{aligned}&\int_\Omega eE^{\alpha\beta\lambda\Gamma}\,(\gamma_{\alpha\beta}(\vec{u}_h)\gamma_{\lambda\Gamma}(\vec{v}_h) + \frac{e^2}{12}\chi_{\alpha\beta}(\vec{u}_h,\underline{\beta}_h)\chi_{\lambda\Gamma}(\vec{v}_h,\underline{\delta}_h))\sqrt{a}\,d\xi^1 d\xi^2\\ &\qquad = \int \vec{p}\vec{v}_h\sqrt{a}\,d\xi^1 d\xi^2 + \int_{\Gamma_1}(\vec{N}.\vec{v}_h - M^\alpha\delta_{\alpha h})\,d\gamma\ ,\quad \forall(\vec{v}_h,\underline{\delta}_h)\in\tilde{\tilde{V}}\ ,\end{aligned}\right\} \tag{7.3.1}$$

where the space $\tilde{\tilde{V}}_h$ is defined as follows :

$$\left.\begin{aligned}\tilde{\tilde{V}}_h = \{(\vec{v}_h,\underline{\delta}_h) &\text{ such that } \vec{v}_h \in V_{h1}\times V_{h1}\times V_{h2}\ ,\quad \underline{\delta}_h \in V_{h1}\times V_{h1}\\ &\text{and } \vec{v}_h,\underline{\delta}_h \text{ satisfy the constraints (7.2.1) to (7.2.3)}\}.\end{aligned}\right\} \tag{7.3.2}$$

The question of existence and uniqueness of a solution for this problem is in progress as well as the study of the effect of the use of numerical integration schemes to compute the integrals in (7.3.1). Meanwhile let us mention that :

i) these methods give good results for some classical bench-marks in thin shell problems : see BERNADOU-MATO EIROA [1972]. Also, see BATOZ-BENTAHAR and DHATT [1982] for other closed related methods ;

ii) KIKUCHI [1981] has proved the convergence for *plate problems* and he has got error estimates such that

$$\sum_{i,j=1}^{2} \|\partial_{hij}u_h - \partial_{ij}u\|_{L^2} \le Ch\,\|f\|_{L^2}\ ,$$

where u and u_h denote respectively the solutions (plate deflections) of the continuous and D.K.T. approximated problems.

7.4. Some examples

7.4.1. Cylindrical roof of SCORDELIS-LO [1964]

This is the roof mentionned in Figure 2.3.1. It is loaded by its own weight ; the shell is simply supported at its extremities by rigid diaphragms and it is free along the longitudinal sides. The weight of the shell is worth 90 ℓb/sq.ft or, equivalently, 0.625 ℓb/sq.in.

By consideration of the symmetries, we just consider a quarter of the domain, i.e., the part ABCD of the roof. We have used the four meshes indicated upon Figure 7.4.1. The results are given upon Figure 7.4.2 to 7.4.5.

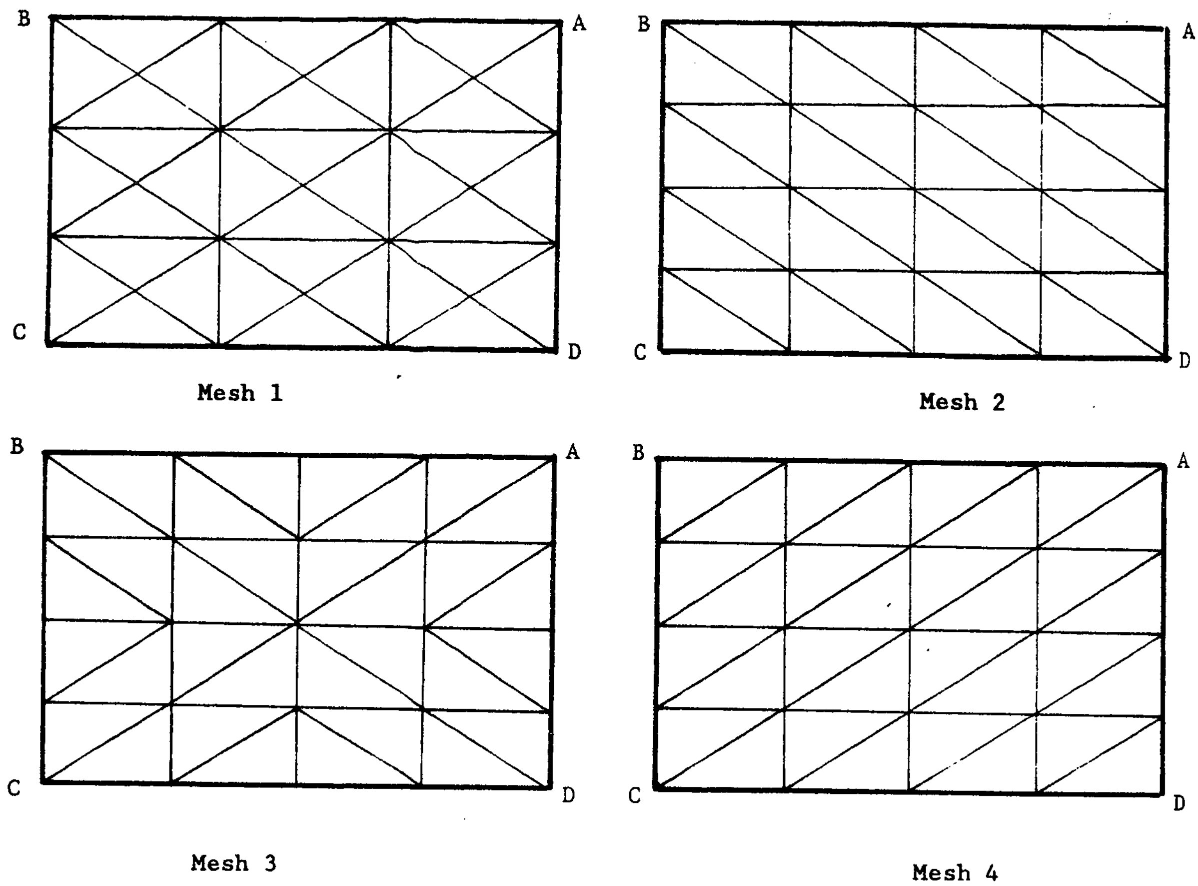


Figure 7.4.1 : The different meshes in use

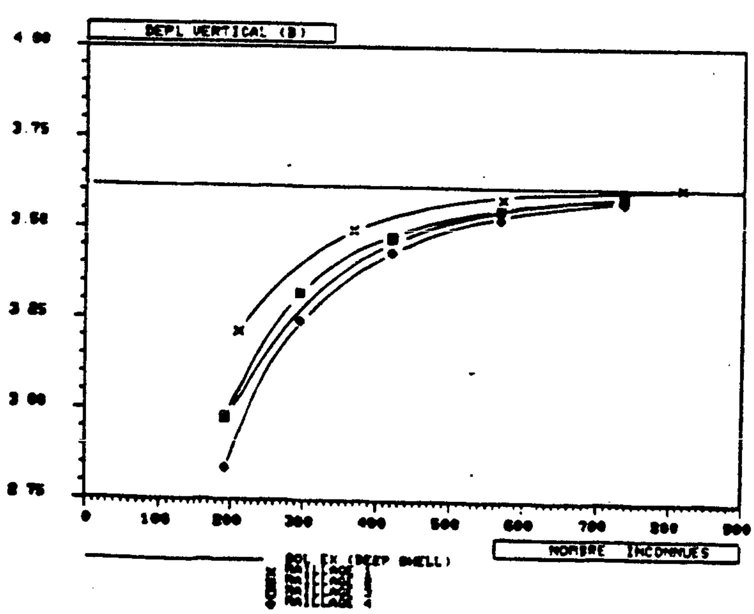


Figure 7.4.2 : Value of the component $u_Z(B)$

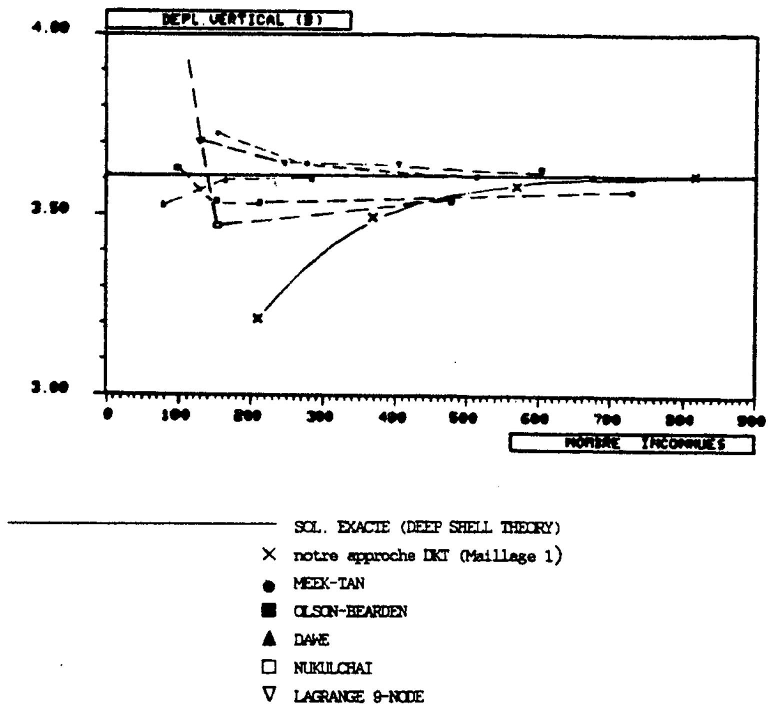


Figure 7.4.3 : Comparison of the values of $u_Z(B)$ as given by different Authors

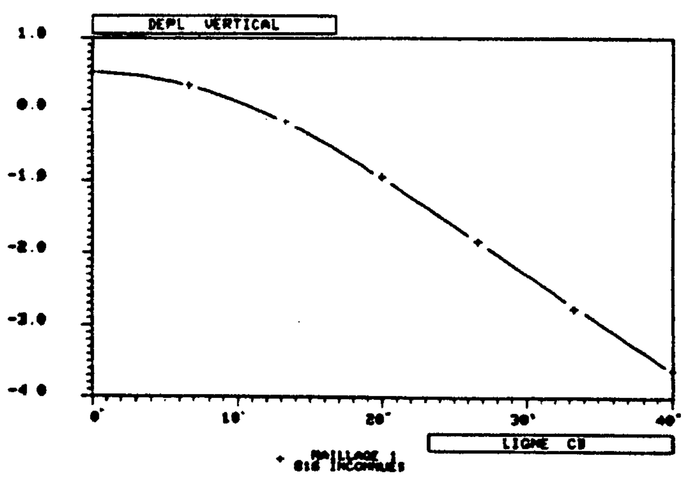


Figure 7.4.4 : Value of u_Z along the side CB

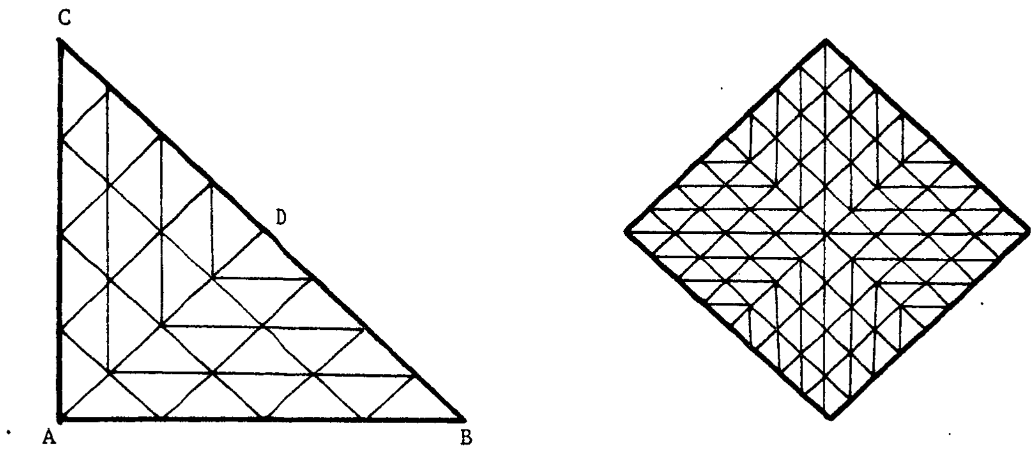
	192 unknowns	210 unknowns	295 unknowns	368 unknowns	420 unknowns	567 unknowns	570 unknowns	736 unknowns	816 unknowns
Mesh 1		0.8900858		0.9680249			0.9921052		1.000975
Mesh 2	0.8240027		0.9196454		0.9630637	0.9835955		0.9939972	
Mesh 3	0.8250166				0.9616398			0.9931191	
Mesh 4	0.7862465		0.8983767		0.9510304	0.9765213		0.9896565	

Figure 7.4.5 : Value of $u_{hZ}(B)/u_Z(B)$ as a function of the mesh in use

7.4.2. Clamped hyperbolic paraboloid under uniform pressure

This example has been considered by CONNOR-BREBBIA [1967], BATOZ [1977], ADINA [1983, ex.A.26], and MEEK-TAN 1986 . The hyperbolic paraboloid is described upon Figure 2.3.3. It is clamped along the entirely of its boundary and it is loaded by a uniform pressure.

The mesh in use is presented upon Figure 7.4.6 and corresponding results are detailed upon Figures 7.4.7 to 7.4.10.



The mesh in use

The entire mesh

Figure 7.4.6 : The meshes

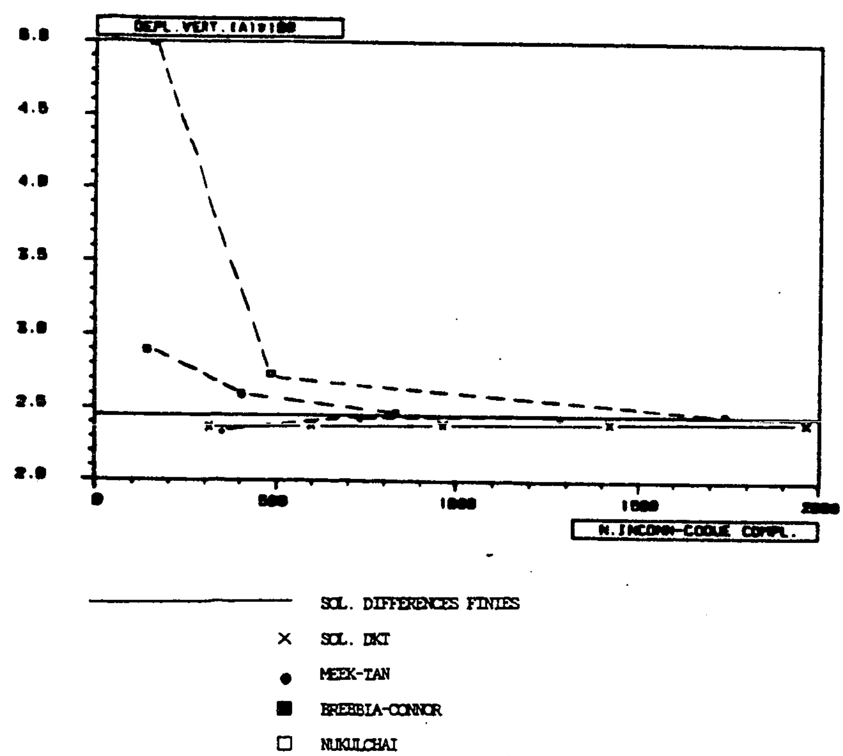


Figure 7.4.7 : Comparison of the values $u_Z(A)$ given by different Authors

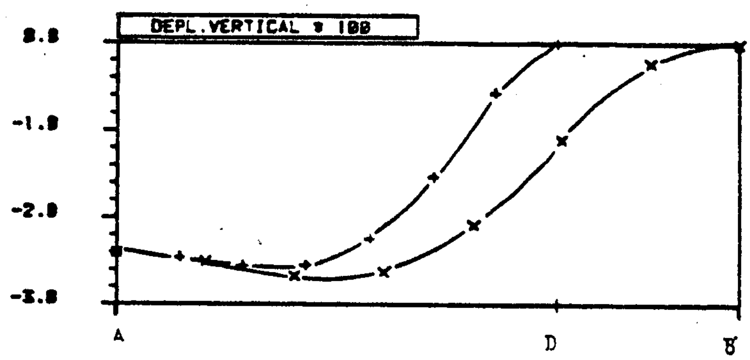


Figure 7.4.8 : Value of 100 × u_Z along AB(×) and AD(+)

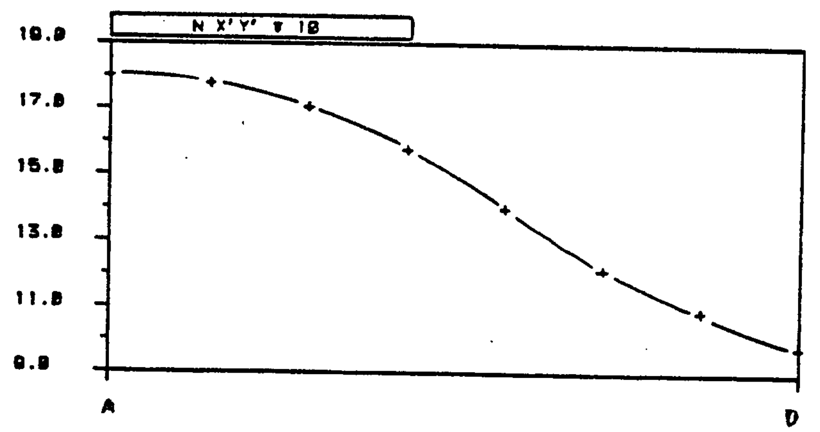


Figure 7.4.9 : Value of 10 × $N_{X'Y'}$ along AD

Number of unknowns for the full shell	317	597	965	1421	1965
Sol_{FEM}/Sol_{FD}	0.9631548	0.9684573	0.9733125	0.9767567	0.9791396

Figure 7.4.10 : Quotient of the values of $u_Z(A)$ computed by FEM or by Finite Differences

PART III : OPTIMAL DESIGN OF A SHELL : COMPUTATION OF THE FUNCTIONAL DERIVATIVE

(extracted from M. BERNADOU, F. PALMA and B. ROUSSELET [1987])

8 - GENERAL PRINCIPLE FOR THE COMPUTATION OF THE FUNCTIONAL DERIVATIVE

8.1. Design variables

Let $\mathscr{C}$ be a thin shell whose middle surface $\bar{S}$ is the image of a plane domain $\bar{\Omega}$ by a regular mapping $\vec{\phi}$ and whose *thickness* e is again a regular mapping defined at any point of $\bar{\Omega}$.

According to the formulations given in paragraph 3, the deformations of the shell $\mathscr{C}$ are characterized by the displacement field $\vec{u}$ of the particles located upon the middle surface of the shell. This field $\vec{u} \in \vec{V}$ is the solution of the variational equation (see (3.4.1))

$$P'(\vec{u}).\vec{v} = 0 \quad , \quad \forall \vec{v} \in \vec{V} \quad , \tag{8.1.1}$$

where $\vec{V}$ is the space of admissible displacements. Subsequently, we will restrict our study to the thin shells whose deformations are governed by the "general" linear equations considered in section 3.5, i.e., with (4.1.1) :

$$a(\vec{u},\vec{v}) = f(\vec{v}) \quad , \quad \forall \vec{v} \in \vec{V} \tag{8.1.2}$$

with

$$a(\vec{u},\vec{v}) = \int_\Omega eE^{\alpha\beta\lambda\mu} \, [\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\mu}(\vec{v}) + \frac{e^2}{12} \bar{\rho}_{\alpha\beta}(\vec{u})\bar{\rho}_{\lambda\mu}(\vec{v})] \sqrt{a} \, d\xi^1 d\xi^2 \tag{8.1.3}$$

$$f(\vec{v}) = \int_\Omega \vec{p}\vec{v} \sqrt{a} \, d\xi^1 d\xi^2 + \int_{\Gamma_1} [\vec{N}\vec{v} + M^\alpha \phi_\alpha] \, d\gamma \tag{8.1.4}$$

and

$$\left.\begin{aligned} \gamma_{\alpha\beta}(\vec{v}) &= \frac{1}{2} (v_{\alpha|\beta}+v_{\beta|\alpha}) - b_{\alpha\beta}v_3 \\ \bar{\rho}_{\alpha\beta}(\vec{v}) &= v_{3|\alpha\beta} - b_\alpha^\lambda b_{\lambda\beta}v_3 + b_\alpha^\lambda v_{\lambda|\beta} + b_\beta^\lambda v_{\lambda|\alpha} + b^\lambda_{\alpha|\beta} v_\lambda \quad . \end{aligned}\right\} \tag{8.1.5}$$

The goal of this study is to propose a methodology of the shape optimization of a general thin elastic shell. The *design variables* are the form of the *middle surface* and the *thickness* of the shell, i.e., $\vec{\phi}$ and e with the notations of sections 2.1 and 2.2. Subsequently, we will set

$$\Phi = (\vec{\phi}, e) \quad . \tag{8.1.6}$$

8.2. The optimization problem

This optimization problem can be formulated as follows :

i) *Minimize a functional* $J_o(\Phi,\vec{u}_\Phi)$ which is explicitely dependent of the design variables Φ and which is dependent of the solution $\vec{u}_\Phi$ of the state equation, i.e., of the solution $\vec{u}_\Phi$ of the equation (8.1.2). Here, we note $\vec{u}_\Phi$ instead of $\vec{u}$ since the solution $\vec{u}$ of equation (8.1.2) is implicitely depending of Φ.

Some examples of such functionals are given by the measures of the weight of the structure, of the stresses, of the critical loads for buckling... ;

ii) *With or without constraints*. These constraints can be bounds on the displacements, on some stresses, on the thickness... They can be also expressed as functionals $J_i(\Phi,\vec{u}_\Phi)$, $i = 1,2,\ldots$.

In order to use the classical optimization algorithms, we will have to compute the derivatives of such functionals $J_o(\Phi,\vec{u}_\Phi)$ and $J_i(\Phi,\vec{u}_\Phi)$ with respect to Φ (see CEA [1986] or ROUSSELET [1986]). Subsequently we will just consider J all that follows can be applied exactly in the same way to any functional $J_i(\Phi,\vec{u}_\Phi)$, $i = 0,1,2,\ldots$.

From the functional J, we can define a new functional j, just depending of Φ, as follows

$$j(\Phi) = J(\Phi,\vec{u}_\Phi) \quad , \tag{8.2.1}$$

and we need to compute the derivative of j with respect to Φ. This is generally difficult since the dependence of $\vec{u}_\Phi$ with respect to Φ is not explicitely known.

By using CEA [1986], this computation can be easily done through the introduction of a new problem : the *adjoint state equation*. Briefly, for a state equation of the form (8.1.2), we will obtain :

$$Dj(\Phi).\Psi = \partial_\Phi J(\Phi,\vec{u}_\Phi).\Psi + \partial_\Phi a(\Phi;\vec{u}_\Phi,\vec{v}_\Phi).\Psi - \partial_\Phi f(\Phi;\vec{v}_\Phi).\Psi \tag{8.2.2}$$

where $\vec{u}_\Phi$ is the solution of the *state equation* (8.1.2), i.e.,

$$a(\Phi;\vec{u}_\Phi,\vec{v}) = f(\Phi;\vec{v}) \quad , \quad \forall\vec{v} \in \vec{V} \tag{8.2.3}$$

and where $\vec{v}_\Phi$ is the solution of the *adjoint state* equation :

$$a(\Phi;\vec{v}_\Phi,\vec{w}) = - \partial_{\vec{u}} J(\Phi,\vec{u}_\Phi).\vec{w} \quad , \quad \forall\vec{w} \in \vec{W} \quad . \tag{8.2.4}$$

Of course the equation (8.2.3) is identical to (8.1.2) but we have explicitely mentionned the dependences of a(.,.) and f(.) with respect to Φ.

Some works have been devoted to this type of problems ; BUDIANSKY-FRANENTHAL-HUTCHINSON [1969], BANICHUK [1975], BOISSERIE-GLOWINSKI [1978], HLAVÁCEK [1983]. Differentiability and computation of differential with respect to midline variations for arches has been adressed in CHENAIS-ROUSSELET [1984] ; the case of midsurface variations is adressed in CHENAIS [1985] and CHENAIS [1987]. The case of shells of revolution is considered in HLAVACEK [1983] and C.A. MOTA SOARES-C.M. MOTA SOARES-MATEUS [1987] ; shallow shells : BANICHUK-LARICHEV [1984].

Thus, in the next paragraphs, we will compute the derivatives of J, a and f with respect to Φ. This will be done for the bilinear form a(.,.) attached to the deformation of a general shell and for some classical linear forms f(.). Finally we will consider some examples of functionals J and we will give their corresponding derivatives with respect to Φ.

9 - COMPUTATION OF THE DERIVATIVE $\partial_\Phi a(\Phi;\vec{u},\vec{v}).\Psi$

With the relation (8.1.3), we obtain

$$a(\Phi;\vec{u},\vec{v}) = \int_\Omega eE^{\alpha\beta\lambda\mu}(\vec{\phi})\ [\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\gamma_{\lambda\mu}(\vec{\phi};\vec{v}) + \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\bar{\rho}_{\lambda\mu}(\vec{\phi},\vec{v})]\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2 \tag{9.1}$$

where we have explicitely mentionned the dependences on $\Phi = (\vec{\phi},e)$. Then, by assuming that the mappings $\vec{\phi}$ and e are sufficiently regular, and by setting

$$\Psi = (\vec{\psi},\epsilon) \quad , \tag{9.2}$$

we obtain

$$\left.\begin{aligned}\partial_\Phi a(\Phi;\vec{u},\vec{v}).\Psi &= \int_\Omega \epsilon E^{\alpha\beta\lambda\mu}(\vec{\phi})\ [\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\gamma_{\lambda\mu}(\vec{\phi};\vec{v}) + \frac{e^2}{4}\,\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})]\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2 \\ &+ \int_\Omega e[\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\gamma_{\lambda\mu}(\vec{\phi};\vec{v}) + \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})]\ [\partial_{\vec{\phi}}\,(E^{\alpha\beta\lambda\mu}(\vec{\phi})\ \sqrt{a(\vec{\phi})}).\vec{\psi}]\ d\xi^1 d\xi^2 \\ &+ \int_\Omega eE^{\alpha\beta\lambda\mu}(\vec{\phi})\ \sqrt{a(\vec{\phi})}\ [\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\partial_{\vec{\phi}}\gamma_{\lambda\mu}(\vec{\phi};\vec{v}).\vec{\psi} + \gamma_{\lambda\mu}(\vec{\phi};\vec{v})\partial_{\vec{\phi}}\gamma_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi} \\ &+ \frac{e^2}{12}\,\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\partial_{\vec{\phi}}\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v}).\vec{\psi} + \frac{e^2}{12}\,\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})\partial_{\vec{\phi}}\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi}]\ d\xi^1 d\xi^2 \quad .\end{aligned}\right\} \tag{9.3}$$

Hence, the computation of $\partial_\Phi a(\Phi;\vec{u},\vec{v}).\Psi$ is reduced to the computation of

$$\partial_{\vec{\phi}}\,(E^{\alpha\beta\lambda\mu}(\vec{\phi})\ \sqrt{a(\vec{\phi})}).\vec{\psi} \tag{9.4}$$

$$\partial_{\vec{\phi}}\,\gamma_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi} \tag{9.5}$$

$$\partial_{\vec{\phi}}\,\rho_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi} \tag{9.6}$$

<u>Computation of</u> $\partial_{\vec{\phi}}\,(E^{\alpha\beta\lambda\mu}(\vec{\phi})\ \sqrt{a(\vec{\phi})}).\vec{\psi}$:

The relation (3.3.2) involves

$$\left.\begin{aligned}\partial_{\vec{\phi}}\,(E^{\alpha\beta\lambda\mu}(\vec{\phi})\ \sqrt{a(\vec{\phi})}).\vec{\psi} &= \frac{E}{2(1+\nu)}\ \sqrt{a(\vec{\phi})}\ [a^{\alpha\lambda}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\beta\mu}(\vec{\phi}).\vec{\psi} + a^{\beta\mu}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\alpha\lambda}(\vec{\phi}).\vec{\psi} \\ &+ a^{\alpha\mu}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\beta\lambda}(\vec{\phi}).\vec{\psi} + a^{\beta\lambda}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\alpha\mu}(\vec{\phi}).\vec{\psi} + \frac{2\nu}{1-\nu}\ a^{\alpha\beta}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\lambda\mu}(\vec{\phi}).\vec{\psi} + \\ &+ \frac{2\nu}{1-\nu}\ a^{\lambda\mu}(\vec{\phi})\ \partial_{\vec{\phi}}a^{\alpha\beta}(\vec{\phi}).\vec{\psi}] + E^{\alpha\beta\lambda\mu}(\vec{\phi})\ \frac{1}{2\sqrt{a(\vec{\phi})}}\ \partial_{\vec{\phi}}\ a(\vec{\phi}).\vec{\psi} \quad .\end{aligned}\right\} \tag{9.7}$$

But

$$a_{\alpha\lambda}(\vec{\phi}) = \vec{a}_\lambda(\vec{\phi}).\vec{a}_\alpha(\vec{\phi}) = \frac{\partial\vec{\phi}}{\partial\xi^\alpha} \cdot \frac{\partial\vec{\phi}}{\partial\xi^\lambda}$$

and hence,

$$\partial_{\vec{\phi}}\, a_{\alpha\lambda}(\vec{\phi}).\vec{\psi} = \vec{a}_\alpha(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^\lambda} + \vec{a}_\lambda(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^\alpha} \qquad (9.8)$$

Thus

$$\frac{1}{2}\,\partial_{\vec{\phi}}a(\vec{\phi}).\vec{\psi} = \{a_{22}(\vec{\phi})\vec{a}_1(\vec{\phi}) - a_{12}(\vec{\phi})\vec{a}_2(\vec{\phi})\}.\frac{\partial\vec{\psi}}{\partial\xi^1} + \{a_{11}(\vec{\phi})\vec{a}_2(\vec{\phi}) - a_{12}(\vec{\phi})\vec{a}_1(\vec{\phi})\}.\frac{\partial\vec{\psi}}{\partial\xi^2} \; .$$

Since

$$\left.\begin{aligned} a_{11}(\vec{\phi}) &= a(\vec{\phi})a^{22}(\vec{\phi}) \;, \\ a_{12}(\vec{\phi}) &= -\,a(\vec{\phi})a^{12}(\vec{\phi}) \;, \\ a_{22}(\vec{\phi}) &= a(\vec{\phi})a^{11}(\vec{\phi}) \;, \end{aligned}\right\} \qquad (9.9)$$

we obtain

$$\frac{1}{2}\,\partial_{\vec{\phi}}a(\vec{\phi}).\vec{\psi} = a(\vec{\phi})\vec{a}^\alpha(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^\alpha} \; . \qquad (9.10)$$

It remains to compute $\partial_{\vec{\phi}}a^{\beta\mu}(\vec{\phi}).\vec{\psi}$. By using relations (9.8) and $a^{\alpha\lambda}(\vec{\phi})a_{\alpha\mu}(\vec{\phi}) = \delta^\lambda_\mu$, we get :

$$\partial_{\vec{\phi}}(a^{\beta\mu}(\vec{\phi})).\vec{\psi} = -\,[a^{\beta\alpha}(\vec{\phi})\vec{a}^\mu(\vec{\phi}) + a^{\mu\alpha}(\vec{\phi})\vec{a}^\beta(\vec{\phi})].\frac{\partial\vec{\psi}}{\partial\xi^\alpha} \; . \qquad (9.11)$$

Finally, we report the relations (9.10) and (9.11) into the expression (9.7) and we obtain :

$$\left.\begin{aligned} \partial_{\vec{\phi}}\,\{E^{\alpha\beta\lambda\mu}(\vec{\phi})\,\sqrt{a(\vec{\phi})}\}.\vec{\psi} = -\,\sqrt{a(\vec{\phi})}\,\frac{\partial\vec{\psi}}{\partial\xi^\nu}\,.\{E^{\nu\beta\lambda\mu}(\vec{\phi})\vec{a}^\alpha(\vec{\phi}) + E^{\alpha\nu\lambda\mu}(\vec{\phi})\vec{a}^\beta(\vec{\phi}) \\ + E^{\alpha\beta\nu\mu}(\vec{\phi})\vec{a}^\lambda(\vec{\phi}) + E^{\alpha\beta\lambda\nu}(\vec{\phi})\vec{a}^\mu(\vec{\phi}) - E^{\alpha\beta\lambda\mu}(\vec{\phi})\vec{a}^\nu(\vec{\phi})\} \; . \end{aligned}\right\} \qquad (9.12)$$

<u>Computation of</u> $\partial_{\vec{\phi}}\gamma_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi}$

We start from the expression

$$\gamma_{\alpha\beta}(\vec{\phi};\vec{u}) = \frac{1}{2}\,(u_{\alpha,\beta}+u_{\beta,\alpha}) - \Gamma^\nu_{\alpha\beta}(\vec{\phi})u_\nu - b_{\alpha\beta}(\vec{\phi})u_3 \; . \qquad (9.13)$$

In this expression, we have explicitely mentionned the dependences in $\vec{\phi}$ of all the parameters. In particular, let us mentionned that *the components* u_1, u_2 *and* u_3 *are considered as independent of* $\vec{\phi}$. Then we obtain

$$\partial_{\vec{\phi}}\gamma_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi} = -\,u_\nu\partial_{\vec{\phi}}(\Gamma^\nu_{\alpha\beta}(\vec{\phi})).\vec{\psi} - u_3\partial_{\vec{\phi}}(b_{\alpha\beta}(\vec{\phi})).\vec{\psi} \; . \qquad (9.14)$$

By definition

$$\Gamma^\nu_{\alpha\beta}(\vec{\phi}) = \vec{a}^\nu(\vec{\phi}).\vec{a}_{\alpha,\beta}(\vec{\phi})$$

so that

$$\partial_{\vec{\phi}}(\Gamma^{\nu}_{\alpha\beta}(\vec{\phi})).\vec{\psi} = \vec{a}_{\alpha,\beta}(\vec{\phi})\partial_{\vec{\phi}}(\vec{a}^{\nu}(\vec{\phi})).\vec{\psi} + \vec{a}^{\nu}(\vec{\phi}).\frac{\partial^2\vec{\psi}}{\partial\xi^{\alpha}\partial\xi^{\beta}} \quad . \tag{9.15}$$

The relations $\vec{a}^{\nu}(\vec{\phi}).\vec{a}_i(\vec{\phi}) = \delta^{\nu}_i$ involve

$$\left.\begin{aligned} &\vec{a}_{\alpha}(\vec{\phi}).(\partial_{\vec{\phi}}(\vec{a}^{\nu}(\vec{\phi})).\vec{\psi}) + \vec{a}^{\nu}(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\alpha}} = 0 \\ &\vec{a}_3(\vec{\phi}).(\partial_{\vec{\phi}}(\vec{a}^{\nu}(\vec{\phi})).\vec{\psi}) + \vec{a}^{\nu}(\vec{\phi}).(\partial_{\vec{\phi}}(\vec{a}_3(\vec{\phi})).\vec{\psi}) = 0 \quad . \end{aligned}\right\} \tag{9.16}$$

But $\vec{a}_3(\vec{\phi}).\vec{a}_{\alpha}(\vec{\phi}) = 0$ and $\vec{a}_3(\phi).\vec{a}_3(\vec{\phi}) = 1$ involve

$$\left.\begin{aligned} &\vec{a}_{\alpha}(\vec{\phi}).\partial_{\vec{\phi}}(\vec{a}_3(\vec{\phi}).\vec{\psi}) = - \vec{a}_3(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\alpha}} \\ &\vec{a}_3(\vec{\phi}).\partial_{\vec{\phi}}(\vec{a}_3(\vec{\phi}).\vec{\psi}) = 0 \end{aligned}\right\}$$

so that

$$\partial_{\vec{\phi}}(\vec{a}_3(\vec{\phi})).\vec{\psi} = - (\vec{a}_3(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\alpha}})\vec{a}^{\alpha}(\vec{\phi}) \quad . \tag{9.17}$$

Substituting (9.17) into (9.16), we obtain

$$\partial_{\vec{\phi}}(\vec{a}^{\nu}(\vec{\phi})).\vec{\psi} = - (\vec{a}^{\nu}(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\mu}})\vec{a}^{\mu}(\vec{\phi}) + a^{\nu\mu}(\vec{\phi})(\vec{a}_3(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\mu}})\vec{a}^{3}(\vec{\phi}) \quad . \tag{9.18}$$

Then, a new substitution into (9.15) and the relation

$$\vec{a}_{\alpha,\beta}(\vec{\phi}) = \Gamma^{\lambda}_{\alpha\beta}(\vec{\phi})\vec{a}_{\lambda}(\vec{\phi}) + b_{\alpha\beta}(\vec{\phi})\vec{a}_3(\vec{\phi}) \tag{9.19}$$

give

$$\partial_{\vec{\phi}}(\Gamma^{\nu}_{\alpha\beta}(\vec{\phi})).\vec{\psi} = \vec{a}^{\nu}(\vec{\phi}).(\frac{\partial^2\vec{\psi}}{\partial\xi^{\alpha}\partial\xi^{\beta}} - \Gamma^{\lambda}_{\alpha\beta}(\vec{\phi})\ \frac{\partial\vec{\psi}}{\partial\xi^{\lambda}}) + a^{\nu\lambda}(\vec{\phi})b_{\alpha\beta}(\vec{\phi})(\vec{a}_3(\vec{\phi}).\frac{\partial\vec{\psi}}{\partial\xi^{\lambda}}) \quad . \tag{9.20}$$

Now, it remains to compute $\partial_{\vec{\phi}}(b_{\alpha\beta}(\vec{\phi})).\vec{\psi}$. By definition

$$b_{\alpha\beta}(\vec{\phi}) = \vec{a}_3(\vec{\phi}).\vec{a}_{\alpha,\beta}(\vec{\phi})$$

so that

$$\partial_{\vec{\phi}}(b_{\alpha\beta}(\vec{\phi})).\vec{\psi} = \vec{a}_3(\vec{\phi}).\frac{\partial^2\vec{\psi}}{\partial\xi^{\alpha}\partial\xi^{\beta}} + \vec{a}_{\alpha,\beta}(\vec{\phi}).(\partial_{\vec{\psi}}(\vec{a}_3(\vec{\phi})).\vec{\psi}) \quad .$$

Then, the relations (9.17) and (9.19) involve

$$\partial_{\vec{\phi}}(b_{\alpha\beta}(\vec{\phi})).\vec{\psi} = \vec{a}_3(\vec{\phi}).(\frac{\partial^2\vec{\psi}}{\partial\xi^{\alpha}\partial\xi^{\beta}} - \Gamma^{\lambda}_{\alpha\beta}(\vec{\phi})\ \frac{\partial\vec{\psi}}{\partial\xi^{\lambda}}) \quad . \tag{9.21}$$

By substituting the relations (9.20) and (9.21) into relation (9.14), we get

$$\partial_{\vec\phi}(\gamma_{\alpha\beta}(\vec\phi;\vec u)).\vec\psi = -\vec u(\vec\phi).(\frac{\partial^2\vec\psi}{\partial\xi^\alpha\partial\xi^\beta} - \Gamma^\lambda_{\alpha\beta}(\vec\phi)\frac{\partial\vec\psi}{\partial\xi^\lambda}) - u_\nu a^{\nu\lambda}(\vec\phi)b_{\alpha\beta}(\vec\phi)(\vec a_3(\vec\phi).\frac{\partial\vec\psi}{\partial\xi^\lambda}) \quad . \qquad (9.22)$$

<u>Computation of</u> $\partial_{\vec\phi}\bar\rho_{\alpha\beta}(\vec\phi;\vec u).\vec\psi$: we will use the expression (8.1.5) for $\bar\rho_{\alpha\beta}(\vec u)$; it can be written as :

$$\left.\begin{aligned} \bar\rho_{\alpha\beta}(\vec\phi;\vec u) &= u_{3,\alpha\beta} - \Gamma^\nu_{\alpha\beta}(\vec\phi)u_{3,\nu} - b^\nu_\alpha(\vec\phi)b_{\nu\beta}(\vec\phi)u_3 + \\ &+ (b^\nu_{\alpha|\beta}(\vec\phi) - b^\lambda_\alpha(\vec\phi)\Gamma^\nu_{\lambda\beta}(\vec\phi) - b^\lambda_\beta(\vec\phi)\Gamma^\nu_{\lambda\alpha}(\vec\phi))u_\nu + b^\nu_\alpha(\vec\phi)u_{\nu,\beta} + b^\nu_\beta(\vec\phi)u_{\nu,\alpha} \quad . \end{aligned}\right\} \qquad (9.23)$$

The relations (9.20) and (9.21) give respectively the derivatives with respect to $\vec\phi$ of the functions $\vec\phi \to \Gamma^\nu_{\alpha\beta}(\vec\phi)$ and $\vec\phi \to b_{\alpha\beta}(\vec\phi)$. Since $b^\nu_\alpha(\vec\phi) = a^{\nu\beta}(\vec\phi)b_{\alpha\beta}(\vec\phi)$, it comes :

$$\partial_{\vec\phi}(b^\nu_\alpha(\vec\phi)).\vec\psi = b_{\alpha\beta}(\vec\phi)\partial_{\vec\phi}(a^{\nu\beta}(\vec\phi)).\vec\psi + a^{\nu\beta}(\vec\phi)\partial_{\vec\phi}(b_{\alpha\beta}(\vec\phi)).\vec\psi$$

and hence, with relations (9.11) and (9.21) :

$$\left.\begin{aligned} \partial_{\vec\phi}(b^\nu_\alpha(\vec\phi)).\vec\psi &= a^{\nu\beta}(\vec\phi)\vec a_3(\vec\phi).[\frac{\partial^2\vec\psi}{\partial\xi^\alpha\partial\xi^\beta} - \Gamma^\lambda_{\alpha\beta}(\vec\phi)\frac{\partial\vec\psi}{\partial\xi^\lambda}] \\ &- [b^\lambda_\alpha(\vec\phi)a^{\nu\beta}(\vec\phi) + b^\beta_\alpha(\vec\phi)a^{\nu\lambda}(\vec\phi)](\vec a_\beta(\vec\phi).\frac{\partial\vec\psi}{\partial\xi^\lambda}) \quad . \end{aligned}\right\} \qquad (9.24)$$

Now, it remains to express $\partial_{\vec\phi}(b^\nu_{\alpha|\beta}(\vec\phi)).\vec\psi$. By definition, we have

$$b^\lambda_\alpha(\vec\phi) = -\vec a^\lambda(\vec\phi).\vec a_{3|\alpha}(\vec\phi) \quad ,$$

so that

$$b^\lambda_{\alpha|\beta}(\vec\phi) = -\vec a^\lambda|_\beta(\vec\phi).\vec a_{3|\alpha}(\vec\phi) - \vec a^\lambda(\vec\phi).\vec a_{3|\alpha\beta}(\vec\phi) \quad .$$

But

$$\vec a^\lambda|_\beta(\vec\phi) = b^\lambda_\beta(\vec\phi)\vec a_3(\vec\phi)$$

$$\vec a_{3|\alpha}(\vec\phi) = \vec a_{3,\alpha}(\vec\phi) = -b^\nu_\alpha(\vec\phi)\vec a_\nu(\vec\phi)$$

$$\vec a_{3|\alpha\beta}(\vec\phi) = \vec a_{3,\alpha\beta}(\vec\phi) - \Gamma^\nu_{\alpha\beta}(\vec\phi)\vec a_{3,\nu}(\vec\phi)$$

so that

$$\left.\begin{aligned} b^\lambda_{\alpha|\beta}(\vec\phi) &= -\vec a^\lambda(\vec\phi).\vec a_{3,\alpha\beta}(\vec\phi) - \Gamma^\nu_{\alpha\beta}(\vec\phi)b^\lambda_\nu(\vec\phi) \\ &= -a^{\lambda\nu}(\vec\phi)\vec a_\nu(\vec\phi).\vec a_{3,\alpha\beta}(\vec\phi) - \Gamma^\nu_{\alpha\beta}(\vec\phi)b^\lambda_\nu(\vec\phi) \quad . \end{aligned}\right\} \qquad (9.25)$$

By taking into account the relations (9.11) (9.20) (9.24), the computation of the derivative with respect to $\vec{\phi}$ of the function $\vec{\phi} \to b^{\lambda}_{\alpha|\beta}(\vec{\phi})$ amounts to compute the derivative of the product $\vec{a}_{\nu}(\vec{\phi}).\vec{a}_{3,\alpha\beta}(\vec{\phi})$. This one can be written as :

$$\vec{a}_{\nu}(\vec{\phi}).\vec{a}_{3,\alpha\beta}(\vec{\phi}) = - \vec{a}_{\nu,\alpha\beta}(\vec{\phi}).\vec{a}_{3}(\vec{\phi}) - \vec{a}_{\nu,\alpha}(\vec{\phi}).\vec{a}_{3,\beta}(\vec{\phi}) - \vec{a}_{\nu,\beta}(\vec{\phi}).\vec{a}_{3,\alpha}(\vec{\phi}) \quad .$$

Then, with the relations of Gauss and Weingarten

$$\vec{a}_{\nu,\alpha}(\vec{\phi}) = \Gamma^{\lambda}_{\nu\alpha}(\vec{\phi})\vec{a}_{\lambda}(\vec{\phi}) + b_{\nu\alpha}(\vec{\phi})\vec{a}_{3}(\vec{\phi})$$

$$\vec{a}_{3,\alpha}(\vec{\phi}) = - b^{\lambda}_{\alpha}(\vec{\phi})\vec{a}_{\lambda}(\vec{\phi})$$

we obtain

$$\vec{a}_{\nu}(\vec{\phi}).\vec{a}_{3,\alpha\beta}(\vec{\phi}) = - \vec{a}_{\nu,\alpha\beta}(\vec{\phi}).\vec{a}_{3}(\vec{\phi}) + \Gamma^{\delta}_{\nu\alpha}(\vec{\phi})b_{\beta\delta}(\vec{\phi}) + \Gamma^{\delta}_{\nu\beta}(\vec{\phi})b_{\alpha\delta}(\vec{\phi}) \tag{9.26}$$

so that with (9.17) (9.20) and (9.21)

$$\left.\begin{aligned}
\partial_{\vec{\phi}}(\vec{a}_{\nu}(\vec{\phi}).\vec{a}_{3,\alpha\beta}(\vec{\phi})).\vec{\psi} = & - \vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\nu\alpha\beta} + (\vec{a}^{\lambda}(\vec{\phi}).\vec{a}_{\nu,\alpha\beta}(\vec{\phi}))(\vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\lambda}) \\
& + b_{\beta\delta}(\vec{\phi})[\vec{a}^{\delta}(\vec{\phi}).(\vec{\psi}_{,\nu\alpha} - \Gamma^{\lambda}_{\nu\alpha}(\vec{\phi})\vec{\psi}_{,\lambda}) + a^{\delta\lambda}(\vec{\phi})b_{\nu\alpha}(\vec{\phi})(\vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\lambda})] \\
& + \Gamma^{\delta}_{\nu\alpha}(\vec{\phi})\vec{a}_{3}(\vec{\phi}).(\vec{\psi}_{,\beta\delta} - \Gamma^{\lambda}_{\beta\delta}(\vec{\phi})\vec{\psi}_{,\lambda}) + b_{\alpha\delta}(\vec{\phi})[\vec{a}^{\delta}(\vec{\phi}).(\vec{\psi}_{,\nu\beta}(\vec{\phi}) - \Gamma^{\lambda}_{\nu\beta}(\vec{\phi})\vec{\psi}_{,\lambda})) \\
& + a^{\delta\lambda}(\vec{\phi})b_{\nu\beta}(\vec{\phi})(\vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\lambda})] + \Gamma^{\delta}_{\nu\beta}(\vec{\phi})\vec{a}_{3}(\vec{\phi}).(\vec{\psi}_{,\alpha\delta} - \Gamma^{\lambda}_{\alpha\delta}(\vec{\phi})\vec{\psi}_{,\lambda})
\end{aligned}\right\}$$

or

$$\left.\begin{aligned}
\partial_{\vec{\phi}}(\vec{a}_{\nu}(\vec{\phi}).\vec{a}_{3,\alpha\beta}(\vec{\phi})).\vec{\psi} = & - \vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\nu\alpha\beta} + \vec{a}^{\delta}(\vec{\phi}).(b_{\beta\delta}(\vec{\phi})\vec{\psi}_{,\nu\alpha} + b_{\alpha\delta}(\vec{\phi})\vec{\psi}_{,\nu\beta}) \\
& + \vec{a}_{3}(\vec{\phi}).(\Gamma^{\delta}_{\nu\alpha}(\vec{\phi})\vec{\psi}_{,\beta\delta} + \Gamma^{\delta}_{\nu\beta}(\vec{\phi})\vec{\psi}_{,\alpha\delta}) + [(\vec{a}^{\lambda}(\vec{\phi}).\vec{\phi}_{,\nu\alpha\beta}) + b^{\lambda}_{\beta}(\vec{\phi})b_{\nu\alpha}(\vec{\phi}) + \\
& + b^{\lambda}_{\alpha}(\vec{\phi})b_{\nu\beta}(\vec{\phi}) - \Gamma^{\delta}_{\nu\alpha}(\vec{\phi})\Gamma^{\lambda}_{\beta\delta}(\vec{\phi}) - \Gamma^{\delta}_{\nu\beta}(\vec{\phi})\Gamma^{\lambda}_{\alpha\delta}(\vec{\phi})](\vec{a}_{3}(\vec{\phi}).\vec{\psi}_{,\lambda}) \\
& + [- b_{\beta\delta}(\vec{\phi})\Gamma^{\lambda}_{\nu\alpha}(\vec{\phi}) - b_{\alpha\delta}(\vec{\phi})\Gamma^{\lambda}_{\nu\beta}(\vec{\phi})(\vec{a}^{\delta}(\vec{\phi}).\vec{\psi}_{,\lambda}) \quad .
\end{aligned}\right\} \tag{9.27}$$

Coming back to (9.25), the relations (9.11) (9.20) (9.24) (9.26) and (9.27) involve

$$\left.\begin{aligned}
&\partial_{\vec\phi}(b^\lambda_{\alpha|\beta}(\vec\phi)).\vec\psi = \\
&- \{(\vec a_3(\vec\phi).\vec\phi_{,\nu\alpha\beta}) - \Gamma^\delta_{\nu\alpha}(\vec\phi)b_{\beta\delta}(\vec\phi) - \Gamma^\delta_{\nu\beta}(\vec\phi)b_{\alpha\delta}(\vec\phi)\}\{(a^{\lambda\epsilon}(\vec\phi)\vec a^\nu(\vec\phi) + (a^{\nu\epsilon}(\vec\phi)\vec a^\lambda(\vec\phi)).\vec\psi_{,\epsilon}\} \\
&+ a^{\lambda\nu}(\vec\phi)\vec a_3(\vec\phi).\vec\psi_{,\nu\alpha\beta} - a^{\lambda\nu}(\vec\phi)\vec a^\delta(\vec\phi).\{b_{\beta\delta}(\vec\phi)\vec\psi_{,\nu\alpha} + b_{\alpha\delta}(\vec\phi)\vec\psi_{,\nu\beta}\} \\
&- a^{\lambda\nu}(\vec\phi)\vec a_3(\vec\phi).\{\Gamma^\delta_{\nu\alpha}(\vec\phi)\vec\psi_{,\beta\delta} + \Gamma^\delta_{\nu\beta}(\vec\phi)\vec\psi_{,\alpha\delta}\} - a^{\lambda\nu}(\vec\phi)[(\vec a^\epsilon(\vec\phi).\vec\phi_{,\nu\alpha\beta}) + b^\epsilon_\beta(\vec\phi)b_{\nu\alpha}(\vec\phi) + b^\epsilon_\alpha(\vec\phi)b_{\nu\beta}(\vec\phi) \\
&\qquad - \Gamma^\delta_{\nu\alpha}(\vec\phi)\Gamma^\epsilon_{\beta\delta}(\vec\phi) - \Gamma^\delta_{\nu\beta}(\vec\phi)\Gamma^\epsilon_{\alpha\delta}(\vec\phi)](\vec a_3(\vec\phi).\vec\psi_{,\epsilon}) \\
&+ a^{\lambda\nu}(\vec\phi)[b_{\beta\delta}(\vec\phi)\Gamma^\epsilon_{\nu\alpha}(\vec\phi) + b_{\alpha\delta}(\vec\phi)\Gamma^\epsilon_{\nu\beta}(\vec\phi)](\vec a_3(\vec\phi).\vec\psi_{,\epsilon}) - \Gamma^\nu_{\alpha\beta}(\vec\phi)a^{\lambda\delta}(\vec\phi)\vec a_3(\vec\phi).[\vec\psi_{,\nu\delta} - \Gamma^\epsilon_{\nu\delta}(\vec\phi)\vec\psi_{,\epsilon}] \\
&+ \Gamma^\nu_{\alpha\beta}(\vec\phi)[b^\epsilon_\nu(\vec\phi)a^{\lambda\delta}(\vec\phi) + b^\delta_\nu(\vec\phi)a^{\lambda\epsilon}(\vec\phi)](\vec a_\delta(\vec\phi).\vec\psi_{,\epsilon}) - b^\lambda_\nu(\vec\phi)\vec a^\nu(\vec\phi).[\vec\psi_{,\alpha\beta} - \Gamma^\epsilon_{\alpha\beta}(\vec\phi)\vec\psi_{,\epsilon}] \\
&- b^\lambda_\nu(\vec\phi)a^{\nu\epsilon}(\vec\phi)b_{\alpha\beta}(\vec\phi)(\vec a_3(\vec\phi).\vec\psi_{,\epsilon})
\end{aligned}\right\}$$

or equivalently :

$$\left.\begin{aligned}
&\partial_{\vec\phi}(b^\lambda_{\alpha|\beta}(\vec\phi)).\vec\psi = a^{\lambda\nu}(\vec\phi)(\vec a_3(\vec\phi).\vec\psi_{,\nu\alpha\beta}) \\
&\quad - a^{\lambda\nu}(\vec\phi)\vec a^\delta(\vec\phi).\{b_{\beta\delta}(\vec\phi)\vec\psi_{,\nu\alpha} + b_{\alpha\delta}(\vec\phi)\vec\psi_{,\nu\beta} + b_{\delta\nu}(\vec\phi)\vec\psi_{,\alpha\beta}\} \\
&\quad - a^{\lambda\nu}(\vec\phi)\vec a_3(\vec\phi).\{\Gamma^\delta_{\nu\alpha}(\vec\phi)\vec\psi_{,\beta\delta} + \Gamma^\delta_{\nu\beta}(\vec\phi)\vec\psi_{,\alpha\delta} + \Gamma^\delta_{\alpha\beta}(\vec\phi)\vec\psi_{,\delta\nu}\} \\
&\quad - (\vec a_3(\vec\phi).\vec\phi_{,\nu\alpha\beta})\{(a^{\lambda\epsilon}(\vec\phi)\vec a^\nu(\vec\phi) + a^{\nu\epsilon}(\vec\phi)\vec a^\lambda(\vec\phi)).\vec\psi_{,\epsilon}\} \\
&\quad + \{\Gamma^\delta_{\nu\alpha}(\vec\phi)b_{\delta\beta}(\vec\phi) + \Gamma^\delta_{\nu\beta}(\vec\phi)b_{\alpha\delta}(\vec\phi) + \Gamma^\delta_{\alpha\beta}(\vec\phi)b_{\delta\nu}(\vec\phi)\} \\
&\qquad\quad \{(a^{\lambda\epsilon}(\vec\phi)\vec a^\nu(\vec\phi) + a^{\nu\epsilon}(\vec\phi)\vec a^\lambda(\vec\phi)).\vec\psi_{,\epsilon}\} \\
&\quad + \{\Gamma^\epsilon_{\delta\alpha}(\vec\phi)b_{\beta\nu}(\vec\phi) + \Gamma^\epsilon_{\delta\beta}(\vec\phi)b_{\alpha\nu}(\vec\phi) + \Gamma^\epsilon_{\alpha\beta}(\vec\phi)b_{\nu\delta}(\vec\phi)\}\{a^{\lambda\delta}(\vec\phi)(\vec a^\nu(\vec\phi).\vec\psi_{,\epsilon}\} \\
&\quad - a^{\lambda\nu}(\vec\phi)(\vec a_3(\vec\phi).\vec\psi_{,\epsilon})\{(\vec a^\epsilon(\vec\phi).\vec\phi_{,\nu\alpha\beta}) + b^\epsilon_\beta(\vec\phi)b_{\nu\alpha}(\vec\phi) + b^\epsilon_\alpha(\vec\phi)b_{\nu\beta}(\vec\phi) \\
&\qquad\quad + b^\epsilon_\nu(\vec\phi)b_{\alpha\beta}(\vec\phi) - \Gamma^\delta_{\nu\alpha}(\vec\phi)\Gamma^\epsilon_{\beta\delta}(\vec\phi) - \Gamma^\delta_{\nu\beta}(\vec\phi)\Gamma^\epsilon_{\alpha\delta}(\vec\phi) - \Gamma^\delta_{\alpha\beta}(\vec\phi)\Gamma^\epsilon_{\nu\delta}(\vec\phi)\} \ .
\end{aligned}\right\} \quad (9.28)$$

Now, we have got all the necessary partial results to compute the derivative in $\vec\phi$ of $\bar\rho_{\alpha\beta}(\vec\phi;\vec u)$. The relation (9.23) involves

$$\left.\begin{aligned}
&\partial_{\vec\phi}\ \bar\rho_{\alpha\beta}(\vec\phi;\vec u).\vec\psi = u_\nu\partial_{\vec\phi}(b^\nu_{\alpha|\beta}(\vec\phi) - b^\lambda_\alpha(\vec\phi)\Gamma^\nu_{\lambda\beta}(\vec\phi) - b^\lambda_\beta(\vec\phi)\Gamma^\nu_{\lambda\alpha}(\vec\phi)).\vec\psi \\
&\quad + u_{\nu,\beta}\partial_{\vec\phi}(b^\nu_\alpha(\vec\phi)).\vec\psi + u_{\nu,\alpha}\partial_{\vec\phi}(b^\nu_\beta(\vec\phi)).\vec\psi - u_3\{b^\nu_\alpha(\vec\phi)\partial_{\vec\phi}(b_{\nu\beta}(\vec\phi)).\vec\psi \\
&\quad + b_{\nu\beta}(\vec\phi)\partial_{\vec\phi}(b^\nu_\alpha(\vec\phi)).\vec\psi\} - u_{3,\nu}\partial_{\vec\phi}(\Gamma^\nu_{\alpha\beta}(\vec\phi)).\vec\psi
\end{aligned}\right\} \quad (9.29)$$

where the derivatives with respect to $\vec\phi$ of the functions $b^\nu_{\alpha|\beta}$, b^ν_α , $b_{\nu\beta}$ and $\Gamma^\nu_{\alpha\beta}$ are respectively given by the relations (9.28) (9.24), (9.21) and (9.20).

To obtain the expression of the derivative of $\partial_\Phi a_\Phi(\vec{u},\vec{v}).\Psi$, we report the expression (9.12) into the relation (9.3) so that we obtain

$$\left.\begin{aligned}
\partial_\Phi a_\Phi(\vec{u},\vec{v}).\Psi = &\int_\Omega \epsilon E^{\alpha\beta\lambda\mu}(\vec{\phi})[\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\gamma_{\lambda\mu}(\vec{\phi};\vec{v}) + \frac{e^2}{4}\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})]\sqrt{a(\vec{\phi})}\, d\xi^1 d\xi^2 \\
&+ \int_\Omega e\frac{\partial\vec{\psi}}{\partial\xi^\nu}[E^{\alpha\beta\lambda\mu}(\vec{\phi})\vec{a}^\nu(\vec{\phi}) - 2E^{\alpha\nu\lambda\mu}(\vec{\phi})\vec{a}^\beta(\vec{\phi}) - 2E^{\alpha\beta\lambda\nu}(\vec{\phi})\vec{a}^\mu(\vec{\phi})] \\
&\qquad [\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\gamma_{\lambda\mu}(\vec{\phi};\vec{v}) + \frac{e^2}{12}\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})]\sqrt{a(\vec{\phi})}\, d\xi^1 d\xi^2 \\
&+ \int_\Omega eE^{\alpha\beta\lambda\mu}(\vec{\phi})[\gamma_{\alpha\beta}(\vec{\phi};\vec{u})\partial_{\vec{\phi}}\gamma_{\lambda\mu}(\vec{\phi};\vec{v}).\vec{\psi} + \gamma_{\lambda\mu}(\vec{\phi};\vec{v})\partial_{\vec{\phi}}\gamma_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi} \\
&\qquad + \frac{e^2}{12}\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u})\partial_{\vec{\phi}}\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v}).\vec{\psi} + \frac{e^2}{12}\bar{\rho}_{\lambda\mu}(\vec{\phi};\vec{v})\partial_{\vec{\phi}}\bar{\rho}_{\alpha\beta}(\vec{\phi};\vec{u}).\vec{\psi}]\sqrt{a(\vec{\phi})}\, d\xi^1 d\xi^2
\end{aligned}\right\} \quad (9.30)$$

where the derivatives in $\vec{\phi}$ of the components $\gamma_{\alpha\beta}$ and $\bar{\rho}_{\alpha\beta}$ are given by the relations (9.22) and (9.29).

10 - COMPUTATION OF $\partial_\Phi f_\Phi(\vec{v}_\Phi).\Psi$

A general enough expression of the work of the external loads is given by relation (8.1.4), i.e.,

$$\left.\begin{aligned}
f_\Phi(\vec{v}) = &\int_\Omega p^i(\Phi)v_i\sqrt{a(\vec{\phi})}\, d\xi^1 d\xi^2 \\
&+ \int_a^b \{N^i(\Phi)v_i + M^\alpha(\Phi)(v_{3,\alpha} + b_\alpha^\nu(\vec{\phi})v_\nu)\}\sqrt{a_{\alpha\lambda}(\vec{\phi})\frac{d\xi^\alpha}{dt}\frac{d\xi^\lambda}{dt}}\, dt
\end{aligned}\right\} \quad (10.0.1)$$

where $t \in [a,b] \to (\xi^1(t),\xi^2(t))$ denotes a parameterization of the loaded part Γ_1 of the boundary $\Gamma = \partial\Omega$ (see Figure 10.0.1).

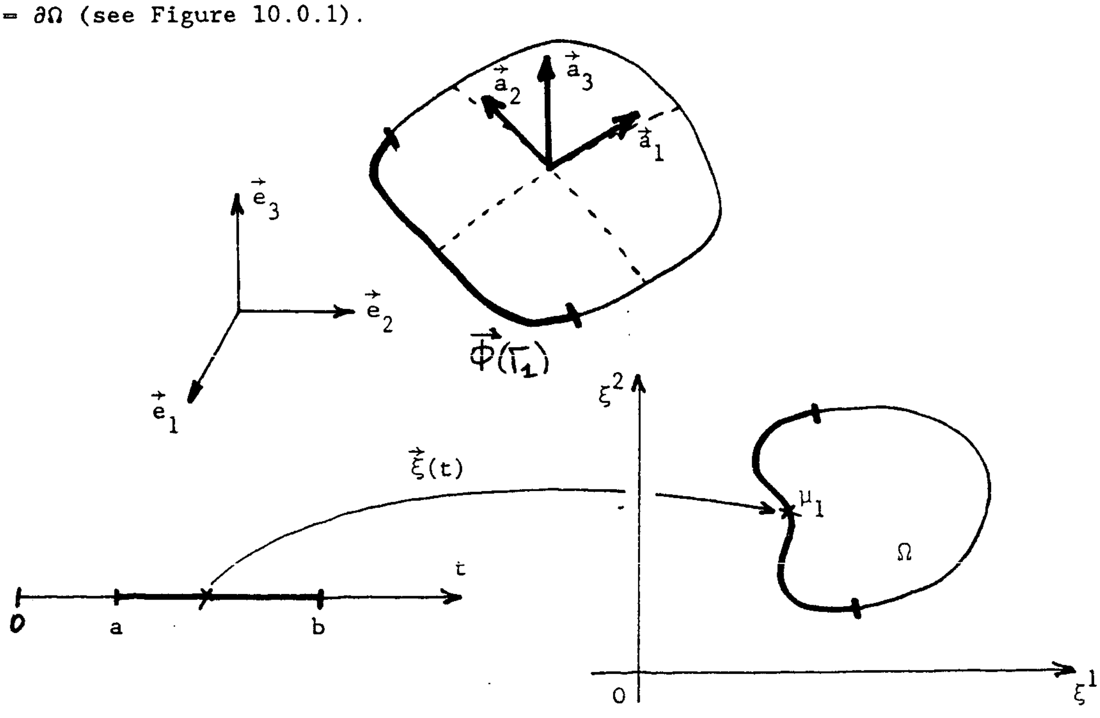


Figure 10.0.1 : Parameterization of the loaded part Γ_1 of the boundary $\Gamma = \partial\Omega$

Subsequently, we consider two kinds of applied loads and we refer to BERNADOU-PALMA and ROUSSELET [1987] for more general cases.

10.1. The pressure loading

By noting $-p\vec{n}(\Phi)$ the surface density of load upon the external loaded surface (associated to $\xi^3 = \frac{e}{2}$), we obtain (BERNADOU-BOISSERIE [1982, (5.2.10)]) :

$$f_\Phi(\vec{v}) = \int_\Omega p[\frac{1}{2}\, e_{,\beta}a^{\alpha\beta}(\vec{\phi})v_\alpha - (1-\frac{1}{2}\, eb^\beta_\beta(\vec{\phi}))v_3]\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \quad . \tag{10.1.1}$$

Then

$$\partial_\Phi f_\Phi(\vec{v}).\Psi = \partial_e f_\Phi(\vec{v}).\epsilon + \partial_{\vec{\phi}} f_\Phi(\vec{v}).\vec{\psi} \tag{10.1.2}$$

where, by using relations (9.10) (9.11) (9.24)

$$\partial_e f_\Phi(\vec{v}).\epsilon = \frac{1}{2}\int_\Omega p[\epsilon_{,\beta}a^{\alpha\beta}(\vec{\phi})v_\alpha + \epsilon b^\beta_\beta(\vec{\phi})v_3]\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \tag{10.1.3}$$

$$\left.\begin{aligned}\partial_{\vec{\phi}} f_\Phi(\vec{v}).\vec{\psi} = \frac{1}{2}\int_\Omega &\{v_\alpha e_{,\beta}[a^{\alpha\beta}(\vec{\phi})\vec{a}^\lambda(\vec{\phi}) - a^{\beta\lambda}(\vec{\phi})\vec{a}^\alpha(\vec{\phi}) - a^{\alpha\lambda}(\vec{\phi})\vec{a}^\beta(\vec{\phi})].\vec{\psi}_{,\lambda}\\ &+ ev_3[a^{\alpha\beta}(\vec{\phi})\vec{a}_3(\vec{\phi}).(\vec{\psi}_{,\alpha\beta}-\Gamma^\lambda_{\alpha\beta}(\vec{\phi})\vec{\psi}_{,\lambda}) - \frac{2}{e}\,(\vec{a}^\lambda(\vec{\phi}).\vec{\psi}_{,\lambda})\\ &+ (b^\beta_\beta(\vec{\phi})\vec{a}^\lambda(\vec{\phi}) - 2b^\lambda_\beta(\vec{\phi})\vec{a}^\beta(\vec{\phi})).\vec{\psi}_{,\lambda}]\}\; p\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \quad .\end{aligned}\right\} \tag{10.1.4}$$

But for very thin shells we can consider the approximation

$$f_\Phi(\vec{v}) = -\int_\Omega pv_3\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \tag{10.1.5}$$

so that

$$\partial_\Phi(f_\Phi(\vec{v})).\psi = -\int_\Omega (\vec{a}^\lambda(\vec{\phi}).\vec{\psi}_{,\lambda})pv_3\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \quad . \tag{10.1.6}$$

Note that in case (10.1.5), the load is independent of the geometry.

10.2. The weighting loads

With BERNADOU-BOISSERIE [1982, (5.1.6)], we obtain

$$f_\Phi(\vec{v}) = \int_\Omega \rho_1 g_o e\; \vec{e}_3.(v_i\vec{a}^i(\vec{\phi}))\sqrt{a(\vec{\phi})}\; d\xi^1 d\xi^2 \quad . \tag{10.2.1}$$

where

$$\left.\begin{aligned}&\rho_1 = \text{mass density}\\ &g_o = \text{gravity acceleration}\\ &\vec{e}_3 = \text{"vertical" vector} \quad .\end{aligned}\right\}$$

□

Thanks to relations (9.10) (9.17) (9.18), we obtain

$$\partial_\Phi f_\Phi(\vec{v}).\psi = \partial_e f_\Phi(\vec{v}).\epsilon + \partial_{\vec{\phi}} f_\Phi(\vec{v}).\vec{\psi} \tag{10.2.2}$$

with

$$\partial_e f_\Phi(\vec{v}).\epsilon = \int_\Omega \rho_1 g_o \epsilon\ v_i(\vec{e}_3.\vec{a}^i(\vec{\phi}))\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2 \tag{10.2.3}$$

$$\left.\begin{aligned}\partial_{\vec{\phi}} f_\Phi(\vec{v}).\vec{\psi} = \int_\Omega \rho_1 g_o e\ \vec{e}_3.\{[a^{\alpha\lambda}(\vec{\phi})(\vec{a}^3(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^3(\vec{\phi})\\ + (\vec{a}^\lambda(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^\alpha(\vec{\phi}) - (\vec{a}^\alpha(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^\lambda(\vec{\phi})]v_\alpha\\ + [(\vec{a}^\lambda(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^3(\vec{\phi}) - (\vec{a}^3(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^\lambda(\vec{\phi})]v_3\}\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2\ .\end{aligned}\right\}$$

The last expression can be reduced to :

$$\left.\begin{aligned}\partial_{\vec{\phi}} f_\Phi(\vec{v}).\vec{\psi} = \int_\Omega \rho_1 g_o e\ \vec{e}_3.\{v_\alpha a^{\alpha\lambda}(\vec{\phi})(\vec{a}^3(\vec{\phi}).\vec{\psi}_{,\lambda})\vec{a}^3(\vec{\phi})\ \sqrt{a(\vec{\phi})}\\ + e^{\lambda\alpha}[v_\alpha\vec{a}_3(\vec{\phi}) - v_3\vec{a}_\alpha(\vec{\phi})] \times \vec{\psi}_{,\lambda}\}\ d\xi^1 d\xi^2\ .\end{aligned}\right\} \tag{10.2.4}$$

11 - EXAMPLE : OPTIMIZATION OF THE WEIGHT OF A SHELL

Orientation

As an example, we consider in this paragraph the optimization of the mass of a shell under some constraints like

- bounds on the thickness,
- bounds on the strain energy.

11.1. The functional giving the weight of the shell

With previous notations, the weight of the shell is given by

$$J(\Phi) = \int_\Omega \rho_1 g_o e\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2 \tag{11.1.1}$$

so that

$$\partial_\Phi J(\Phi).\psi = \int_\Omega \rho_1 g_o \epsilon\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2 + \int_\Omega \rho_1 g_o e(\partial_{\vec{\phi}} \sqrt{a(\vec{\phi})}).\vec{\psi}\ d\xi^1 d\xi^2\ .$$

But relation (9.10) involves

$$\partial_{\vec{\phi}} \sqrt{a(\vec{\phi})}.\vec{\psi} = \sqrt{a(\vec{\phi})}(\vec{a}^\alpha.\frac{\partial\vec{\psi}}{\partial\xi^\alpha})$$

so that

$$\partial_\Phi J_1(\Phi).\Psi = D_\Phi J_1(\Phi).\Psi = \int_\Omega \rho_1 g_o(\epsilon + e\vec{a}^\alpha.\frac{\partial\vec{\psi}}{\partial\xi^\alpha})\ \sqrt{a(\vec{\phi})}\ d\xi^1 d\xi^2\ . \tag{11.1.2}$$

11.2. Some constraints

Of course the minimization of $J(\Phi)$ without any additional constraints would give $e = 0$ as a solution... which is not a realistic one ! So, we will impose simultaneously the two following constraints :

Bounds on the thickness

$$0 < \underline{e} \leq e(\xi^1,\xi^2) \leq \overline{e} \tag{11.2.1}$$

where $\overline{e} > \underline{e} > 0$ are given constants. When one of these constraints becomes saturated the corresponding functional is zero, i.e., $J_1(\Phi) = 0$ or $J_2(\Phi) = 0$ with

$$J_1(\Phi) = e - \underline{e} \quad \text{and} \quad J_2(\Phi) = \overline{e} - e \quad . \tag{11.2.2}$$

For the implementation of the optimization algorithms we will use the derivatives of these functionals, i.e.,

$$\partial_\Phi J_1(\Phi).\Psi = \partial_e J_1(\Phi).\epsilon = \epsilon \tag{11.2.3}$$

or

$$\partial_\Phi J_2(\Phi).\Psi = \partial_e J_2(\Phi).\epsilon = -\epsilon \tag{11.2.4}$$

Bounds of the strain energy

There is a narrow connection between prescribing a bound on the strain energy and prescribing a $L^2(\Omega)$ - bound on the stresses. This functional is currently used in structural optimization as can be noted in the books of BANICHUK [1983], HAUG-CHOI-KOMKOV [1986], PRAGER [1974].

We will impose a constraint of type

$$\frac{1}{2}\, a(\Phi;\vec{u},\vec{u}) \leq C \tag{11.2.5}$$

where the constant C has to be chosen so that

$$\frac{1}{2}\, a(\Phi;\vec{u}_\Phi,\vec{u}_\Phi) < C \tag{11.2.6}$$

since $a(\Phi;\vec{u}_\Phi,\vec{u}_\Phi)$ realizes the minimum of this strain energy. When the constraint (10.2.5) becomes saturated, the corresponding functional is zero, i.e., $J_3(\Phi,\vec{u}) = 0$ with

$$J_3(\Phi,\vec{u}) = C - \frac{1}{2}\, a_\Phi(\vec{u},\vec{u}) \quad . \tag{11.2.7}$$

The derivative with respect to Φ of the functional $\Phi \rightarrow j_3(\Phi) = J_3(\Phi,\vec{u}_\Phi)$ is computed according to (8.2.2). We obtain

$$\partial_\Phi J_3(\Phi,\vec{u}_\Phi).\Psi = -\frac{1}{2}\,\partial_\Phi a_\Phi(\vec{u}_\Phi,\vec{u}_\Phi).\Psi$$

so that

$$\partial_\Phi j_3(\Phi).\Psi = -\frac{1}{2}\,\partial_\Phi a_\Phi(\vec{u}_\Phi,\vec{u}_\Phi).\Psi + \partial_\Phi a_\Phi(\vec{u}_\Phi,\vec{v}_\Phi).\Psi - \partial_\Phi f_\Phi(\vec{v}_\Phi).\Psi \quad . \tag{11.2.8}$$

But with the definition (8.2.4) of the adjoint problem we obtain

$$-\, a(\Phi;\vec{u}_\Phi,\vec{w}) + a(\Phi;\vec{v}_\Phi,\vec{w}) = 0 \quad , \quad \forall\vec{w}\in\vec{W}$$

so that $\vec{v}_\Phi = \vec{u}_\Phi$. Then, the derivative (11.2.8) can be written

$$\partial_\Phi j_3(\Phi).\Psi = \frac{1}{2}\,\partial_\Phi a_\Phi(\vec{u}_\Phi,\vec{u}_\Phi).\Psi - \partial_\Phi f_\Phi(\vec{u}_\Phi).\Psi \quad . \tag{11.2.9}$$

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